Journal of Scientific & Industrial Research
Vol. 84, January 2025, pp. 99-106
DOI: 10.56042/jsir.v84i1.8044

New Approach to Design Optimal Robust Controller for a 2-D Discrete System

Govind Prasad Pandiya[1], Abhay Vidyarthi[2]* & Amrita Pahadia[1]

[1]Department of Electronics & Communication Engineering, Lakshmi Narain College of Technology, Bhopal 462 021, Madhya Pradesh, India

[2]School of Electrical & Electronics Engineering, VIT Bhopal University, Sehore 466 114, Bhopal, Madhya Pradesh, India

Received 13 January 2024; revised 09 September 2024; accepted 12 November 2024

This paper investigates the problem of ensuring the stability of an uncertain system using the unsymmetric Lyapunov function for the two-dimensional discrete system as represented by the Roesser model using the LMI approach. By employing a two-dimensional unsymmetric Lyapunov function, novel LMIs have been developed to ensure stability. The key finding of the present investigation is employing an unsymmetrical Lyapunov matrix for ensuring the stability of a two-dimensional discrete Roesser model, which is a more generalized approach to guarantee the stability of any system. This address the issues of norm-bounded parameter uncertainties, calculate the cost function using an unsymmetric Lyapunov function, and finally design the guaranteed cost controller via a static state feedback technique that not only ascertains the stability of the system but also guarantees an adequate level of performance. The advantages of this newly proposed technique are that it is LMI solvable and numerically tractable. The stability criteria have been checked and ensured based on newly developed stability conditions by considering several examples demonstrating the results effectiveness and supremacy over the previously reported techniques.

Keywords: Guaranteed cost control, Linear matix inequality, Robust analysis, Static state feedback, Unsymmetric lyapunov matrix

Introduction

In recent years, there has been unprecedented growth in studying (2-D) two-dimensional discrete systems. One fundamental characteristic of these systems is information propagation in two distinct and independent directions.[1,2] Further to mention that these systems play a vital role in many distinct fields, including multidimensional digital filtering, signal processing, process control, linear image processing, iterative learning control, and repetitive process control.[3–8] Due to all these aforementioned reasons, the analysis of various two-dimensional discrete systems has been extensively investigated for various new applications.

The stability analysis of these 2-D systems has gained a lot of attention nowadays as many factors affect the stability of these systems, due to which these two-dimensional discrete systems may lead to poor performance. Therefore, a significant investigation has been carried out regarding the stability analysis of these 2-D discrete linear,[1,4–6] as well as non-linear systems.[9–11] There are various 2-D discrete models named FM first and second model,[12,13] Kurek model[14] and Roesser model.[15]

Literature Review

Robust control theory aims to design control systems insensitive to uncertainties and disturbances in the system's dynamics. As it is known, many real-world practical applications may be prone to be affected by uncertainties, which may lead to the poor performance of the system.[1,16] Recent research in the field of medical science also advocates the use of image processing-based algorithms and tools. Recently, most of the study and investigation has been carried out on applications and studies based on image processing and Computer Vision.[17] One of the popular 2-D discrete models which is mostly used in image processing applications is the Roesser model.[15] The 2-D Roesser model has been the subject of extensive research in recent years, focusing on its robustness properties.[3] The Roesser model consists of two state variables, which can be interpreted as the values of the system at different spatial and temporal locations. It is a linear model, which makes it relatively easy to analyze and design controllers. Therefore, this research article is based on the stability analysis of this 2-D discrete Roesser model.

*Author for Correspondence
E-mail: abhay.vidyarthi@vitbhopal.ac.in

Most of the reported work related to the stability analysis for this model has been carried out using the symmetric Lyapunov matrix function.[1,3,5]

It has been observed that if the stability analysis using the symmetric Lyapunov function fails to ensure the stability of the systems, then the stability analysis of such systems can also be as certained using the unsymmetrical Lyapunov function.[18] So, the unsymmetrical Lyapunov function gives a more generalized way of ensuring the stability of any system, and many relevant areas of 2-D discrete systems advocate the usage of such a generalized Lyapunov. To the best of the author's knowledge, until now, nobody has investigated the stability analysis for this model using the unsymmetrical Lyapunov function due to the computational complexity, and this motivated to carry out this work. In this paper has tried to ensure the stability of an uncertain 2-D discrete Roesser model using the unsymmetrical Lyapunov function and designing a guaranteed cost controller using the same.

The unsymmetric Lyapunov function gives a more generalized and wider way of ensuring the stability of the 2-D system under consideration, as claimed.[18] Recently, the system performance has been investigated using the GCC (guaranteed cost control) approach. GCC approach minimizes an upper bound on a quadratic cost function and takes care that the specific upper bound of the cost function will never exceed a particular value for all admissible uncertainties, so if this use this approach to develop a GCC-based controller for an uncertain 2-D Roesser model,[1] then this may help us in not only ensuring the stability but it also in doing the robustness analysis using this more generalized approach which may lead to improved system performance.

This paper aimed to design a guaranteed cost controller using an unsymmetrical Lyapunov function for a 2-D discrete Roesser model. The idea is to find a set of design parameters that guarantees the asymptotic stability of the closed-loop system. This formulation is transformed into an LMI problem, enabling efficient solutions through various numerical optimization techniques.

Materials and Methods

Let us contemplate a Roesser Model describing a two-dimensional discrete linear system that is subject to uncertainty.

$$\begin{bmatrix} X_\theta^h(k,l+1) \\ X_\theta^v(k+1,l) \end{bmatrix} = \left(\underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{14} & A_{15} & A_{16} \\ A_{17} & A_{18} & A_{19} \end{bmatrix}}_{\bar{A}} + \Delta A \right) \underbrace{\begin{bmatrix} X_\theta^h(k,l) \\ X_\theta^v(k,l) \end{bmatrix}}_{X_\theta(k,l)} + \left(\underbrace{\begin{bmatrix} B_{11} \\ B_{12} \\ B_{13} \end{bmatrix}}_{\bar{B}} + \Delta B \right) u(k,l) \quad \ldots (1)$$

where, $X_\theta^h(k,l) \in \mathbb{R}^n$, $X_\theta^v(k,l) \in \mathbb{R}^m$ are horizontal state vector and vertical state vector, respectively $u(k,l) \in \mathbb{R}^q$ Correspond to the system's dimensions. $\bar{A}$ represents a constant matrix of known values that the nominal plant and has dimensions of $\mathbb{R}^{(n+m)\times(n+m)}$. The matrices $\triangle A$ and $\triangle B$ are associated with parameter uncertainties under the assumption that they have a specific form.

$$[\triangle A \;\; \triangle B] = L\,F(k,l)[M_1 M_2] \quad \ldots (2)$$

The matrices L, M_1, and M_2 are known parameters that represent the structural uncertainties in the system, with dimensions ($\mathbb{R}^{(n+m)\times g}$, $\mathbb{R}^{\mathfrak{h}\times(n+m)}$), and $\mathbb{R}^{\mathfrak{h}\times q}$, respectively. On the other hand, the matrix $F(k,l) \in \mathbb{R}^{g\times\mathfrak{h}}$ Represents the unknown parameter uncertainty and fulfils a specific condition.

$$F^T F \leq I \quad \ldots (3)$$

The uncertainties in Eqs (2) and (3) are widely used to filter and design robust control for uncertain systems, making it acceptable to use Eq. (3) without loss of generality. If the system Eq. (2) has a finite set of initial conditions, then there exist two positive integers, denoted as $\mathfrak{r}_1$ and $\mathfrak{r}_2$, that satisfy the following conditions.[1]

$$X_\theta^h(k,0) = 0, k \geq \mathfrak{r}_1; X_\theta^v(0,l) = 0, l \geq \mathfrak{r}_2 \quad \ldots (4)$$

and the initial condition is arbitrary but belong

$$S = \{X_\theta^h(k,0), X_\theta^v(0,j): X_\theta^h(k,0) = ZN_1, X_\theta^v(0,l), ZN_2, N_k^T N_k < 1, k = 1,2,3\} \quad \ldots (5)$$

The structure of initial conditions similar to that in Eq. (4) has been widely used. Note that by selecting Z appropriately, which is a known matrix, it is always possible to restrict the vector. $N_i^T N_i < 1$ $(t = 1,2,3)$ to N_k $(t = 1,2,3)$. This implies that choosing initial conditions in the form of Eq. (5) does not result in any loss of generality.

The uncertain system Eq. (1) is associated with a cost function

$$Ɉ = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} u^T(k,l)Ru(k,l) + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} X_{\theta}^T(k,l)W_1 \quad \ldots (6)$$

In this context, T represents the transpose operation. And $0 < R = R^T \in \mathbb{R}^{q\times q}, 0 < W_1 = W_1^T \in \mathbb{R}^{(n+m)\times(n+m)}$

Definition 1. Consider the uncertain system (1) is examined as quadratically stable, which is a block-diagonal matrix.

$$P_{\theta} = P_{\theta}^{h} \oplus P_{\theta}^{v} = \begin{bmatrix} P_{\theta}^{h} & 0 \\ 0 & P_{\theta}^{v} \end{bmatrix} = \left(\frac{P_{\theta}+P_{\theta}^T}{2}\right) > 0 \quad \ldots (7)$$

where, $P_{\theta}^{h} \in \mathbb{R}^{n\times n}$ and $P_{\theta}^{v} \in \mathbb{R}^{m\times m}$ such that

$$Ⱶ = (A + \triangle A)^T\left(\frac{P_{\theta}+P_{\theta}^T}{2}\right)(A + \triangle A) - \left(\frac{P_{\theta}+P_{\theta}^T}{2}\right) > 0 \quad \ldots (8)$$

Corollary 1. Given the PDSM (positive definite symmetric matrices) $W_{0_1} \in \mathbb{R}^{n\times n}$ and $W_{1_0} \in \mathbb{R}^{n\times n}$ with $W_{0_1} + W_{1_0} = I_m \oplus I_n$, system (1) is asymptotically stable if there exists a PDSM $\boldsymbol{P}_{\theta} = \boldsymbol{P}_{\theta}^{T/2}\boldsymbol{P}_{\theta}^{1/2} \in \mathbb{R}^{n\times n}$, not necessarily symmetric, such that

$$Q = \begin{bmatrix} \begin{bmatrix} \left(\frac{P_{\theta}+P_{\theta}^T}{2}\right)^{\frac{T}{2}} W_{0_1}\left(\frac{P_{\theta}+P_{\theta}^T}{2}\right)^{\frac{1}{2}} & 0 \\ 0 & \left(\frac{P_{\theta}+P_{\theta}^T}{2}\right)^{\frac{T}{2}} W_{1_0}\left(\frac{P_{\theta}+P_{\theta}^T}{2}\right)^{\frac{1}{2}} \end{bmatrix} & AP_{\theta}^T \\ -P_{\theta}^T A & \left(\frac{P_{\theta}+P_{\theta}^T}{2}\right) \end{bmatrix} > 0 \quad \ldots (9)$$

Lemma 1. Given matrices $A \in \mathbb{R}^{n\times n}, H \in \mathbb{R}^{n\times k}, \hat{E} \in \mathbb{R}^{\ell\times n}$ and $Q = Q^T \in \mathbb{R}^{n\times n}$. In that case, there exists a positive definite matrix. P_{θ} to satisfy the equation below

$$[A + \Delta A + H\,FE]^T P_{\theta}\,[A + H\,FE] - Q < 0 \quad \ldots (10)$$

$$[A + H\,FE]^T\left(\frac{P_{\theta}+P_{\theta}^T}{2}\right)[A + H\,FE] - Q > 0 \quad \ldots (11)$$

In the following condition, a scalar $\varepsilon > 0$ exists if and only if the following condition holds for all F satisfying. $F^T F \le I$

$$\begin{bmatrix} \left(\frac{P_{\theta}+P_{\theta}^T}{2}\right) + \varepsilon HH^T & A \\ A^T & \varepsilon^{-1}E^T E - Q \end{bmatrix} > 0 \quad \ldots (12)$$

Subsequently, the aim is to establish a connection between the existence of a quadratic guaranteed cost control matrix and a quadratic stability analysis of the system.

Definition 2. To ensure robust stability for the uncertain system Eq. (1), the cost function Eq. (6) is designed with a quadratic cost function

$$ⴖ = (A + \triangle A)^T\left(\frac{P_{\theta}+P_{\theta}^T}{2}\right)(A + \triangle A) - \left(\frac{P_{\theta}+P_{\theta}^T}{2}\right) + W_1 \le 0 \quad \ldots (13)$$

The following condition holds for all values of $\|F(k,l)\|$. That is less than or equal to 1.

where W_1 is a positive definite matrix that satisfies the property of being symmetric $W_1 = W_1^T$

Lemma 2. A state space model (1) holds for an uncertain system that represents the existence of a quadratic guaranteed cost control matrix, which is associated with initial condition and cost function, respectively Eqs. (5) and (6). denoted by $P_{\theta} = P_{\theta}^T = diag\{P_{\theta}^{h}, P_{\theta}^{v}\} = \left(\frac{P_{\theta}+P_{\theta}^T}{2}\right) > 0$, The control system is considered robust and stable if it encounters the two important criteria:

(i) Quadratic stability, meaning that a positive-definite quadratic Lyapunov function can prove the stability of the closed-loop system.
(ii) The cost function remains bounded in the presence of uncertainties, which are achieved through an appropriate feedback control design that compensates for the uncertainties.

$$Ɉ_{0_1} < \left[(\varrho_1 - 1)\lambda_{max}\left(M^T\left(\frac{P_{\theta}^{h}+P_{\theta}^{h\,T}}{2}\right)M\right) + (\varrho_2 - 1)\lambda_{max}\left(M^T\left(\frac{P_{\theta}^{v}+P_{\theta}^{v\,T}}{2}\right)M\right)\right] \quad \ldots (14)$$

where, λ_{max} Denotes the maximum eigenvalue.

Proof: It can be verified that system formation (1) can be directly deduced from definitions 1 and 2. To prove this, utilized a quadratic two-dimensional Lyapunov matrix [1].

$$\triangle V(X_{\theta}(k,l)) = X_{\theta}^T(k,l)P_{\theta}X_{\theta}(k,l) \quad \ldots (15)$$

This denote $\triangle \vee (X_\theta(k,l))$ as the variation of the Lyapunov function concerning the uncertain parameters $X_\theta(k,l)$.

$$\triangle \vee \left(X_\theta(k,l)\right) = \vee\left(\begin{bmatrix} X_\theta^{h}(k,l+1) \\ X_\theta^{v}(k+1,l)\end{bmatrix}\right) - \vee\left(\underbrace{\begin{bmatrix} X_\theta^{h}(k,l) \\ X_\theta^{v}(k,l)\end{bmatrix}}_{X_\theta(k,l)}\right) \quad \dots (16)$$

Eq. (8) takes the form in view of Eq. (1):

$$\triangle \vee (X_\theta(k,l)) = X_\theta^T(k,l) \text{Ⱶ} X_\theta(k,l) \quad \dots (17)$$

where, Ⱶ is defined in Eq. (8). As the matrix P_θ is quadratic and provides a guaranteed cost, this can infer from definition 2 that is given by

$$X_\theta^T(k,l)(\text{Ⱶ} + W_1)X_\theta(k,l) < 0 \quad \dots (18)$$

From (17) and (18), this obtain

$$\Delta \vee (k,l) + X_\theta^T(k,l)W_1X_\theta(k,l) < 0 \quad \dots (19)$$

Summing over up (16) over $k,l = 0 \to \infty$ yields

$$Ɉ_{0_1} < -\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \triangle \vee (k,l)$$

Furthermore, this can establish positive integers $\ell_1 > 0, \ell_2 > t$ the following hold

$$\sum_{k=0}^{\ell_1}\sum_{l=0}^{\ell_2} \triangle \vee (k,l) = \sum_{l=0}^{\ell_2}\left[X_\theta^{h^T}(\ell_1+1,l)\left(\frac{P_\theta^h + P_\theta^{h^T}}{2}\right)X_\theta^h(\ell_1+1,l) - X_\theta^{h^T}(0,l)\left(\frac{P_\theta^h + P_\theta^{h^T}}{2}\right)X_\theta^h(0,l)\right] + \sum_{k=0}^{\ell_1}\left[X_\theta^{v^T}(k,\ell_2+1)\left(\frac{P_\theta^v + P_\theta^{v^T}}{2}\right)X_\theta^v(k,\ell_2+1) - X_\theta^{v^T}(k,0)\left(\frac{P_\theta^v + P_\theta^{v^T}}{2}\right)X_\theta^v(k,0)\right]$$

And

$$-\sum_{k=0}^{\ell_1}\sum_{l=0}^{\ell_2} \triangle \vee (k,l) = -\sum_{l=0}^{\ell_2}\left[X_\theta^{h^T}(\ell_1+1,l)\left(\frac{P_\theta^h + P_\theta^{h^T}}{2}\right)X_\theta^h(\ell_1+1,l) - X_\theta^{h^T}(0,l)\left(\frac{P_\theta^h + P_\theta^{h^T}}{2}\right)X_\theta^h(0,l)\right] - \sum_{k=0}^{\ell_1}\left[X_\theta^{v^T}(k,\ell_2+1)\left(\frac{P_\theta^v + P_\theta^{v^T}}{2}\right)X_\theta^v(k,\ell_2+1) - X_\theta^{v^T}(k,0)\left(\frac{P_\theta^v + P_\theta^{v^T}}{2}\right)X_\theta^v(k,0)\right]$$

$$< \sum_{l=0}^{\ell_2} X_\theta^{h^T}(0,l)\left(\frac{P_\theta^h + P_\theta^{h^T}}{2}\right)X_\theta^h(0,l) + \sum_{k=0}^{\ell_1} X_\theta^{v^T}(k,0)\left(\frac{P_\theta^v + P_\theta^{v^T}}{2}\right)X_\theta^v(k,0)$$

$$\leq \sum_{l=0}^{\varrho_1-1} X_\theta^{h^T}(0,l)\left(\frac{P_\theta^h + P_\theta^{h^T}}{2}\right)X_\theta^h(0,l) + \sum_{k=0}^{\varrho_2-1} X_\theta^{v^T}(k,0)\left(\frac{P_\theta^v + P_\theta^{v^T}}{2}\right)X_\theta^v(k,0)$$

$$< (\varrho_1 - 1) \curlywedge_{max}\left(\mathcal{M}^T\left(\frac{P_\theta^h + P_\theta^{h^T}}{2}\right)\mathcal{M}\right) + (\varrho_2 - 1) \curlywedge_{max}\left(\mathcal{M}^T\left(\frac{P_\theta^v + P_\theta^{v^T}}{2}\right)\mathcal{M}\right) \quad \dots (20)$$

From Eq. (19) and Eq. (20) this have

$$Ɉ_{0_1} < \left[(\varrho_1 - 1) \curlywedge_{max}\left(\mathcal{M}^T\left(\frac{P_\theta^h + P_\theta^T}{2}\right)\mathcal{M}\right) + (\varrho_2 - 1) \curlywedge_{max}\left(\mathcal{M}^T\left(\frac{P_\theta^v + P_\theta^T}{2}\right)\mathcal{M}\right)\right] \quad \dots (21)$$

To prove this, this has utilized Eqs (21), (4), and (5), along with the fact that the limit of $\lim_{i+j\to\infty} X_\theta(k,l) = 0$ approaches zero as $k + l$ approaches infinity; with this, this have completed the proof.

Lemma 3[1]. Suppose this have real matrices M, L and Q of appropriate dimensions, where $M = M^T$ and $Q = Q^T > 0$, then

$$\begin{bmatrix} M & L^T \\ L & Q^{-1}\end{bmatrix} < 0 \text{ or equivalently } \begin{bmatrix} Q^{-1} & L \\ L^T & \mathcal{M}\end{bmatrix} < 0,$$

If and only if $Q < 0, M - LQ^{-1}L^T < 0,$ … (22)

In the view of the Schur complement Eq. (22) is the same as Eq. (9).

Theorem 1[18]. If there exist matrices P_θ^h and P_θ^v of appropriate dimension, not necessarily symmetric, such that P_θ can be expressed as the direct sum of P_θ^h and P_θ^v, then the system (1) exhibits asymptotic stability.

$$\begin{bmatrix} \left(\frac{P_\theta + P_\theta^T}{2}\right) & -A^T P_\theta \\ -P_\theta{}^T A & \left(\frac{P_\theta + P_\theta^T}{2}\right)\end{bmatrix} > 0 \quad \dots (23)$$

Theorem 2. For a positive scalar, the system (5) with zero initial conditions has a guaranteed cost function

Ɉ_{0_1}, if there existsa symmetric matrix $S > 0$, and matrixP_θ not necessarily symmetric, such that the following LMI holds:

$$\begin{bmatrix} -S & \text{A}S & L & 0 & 0 \\ S\text{A}^T & -S & 0 & SM_1^T & SW_1^{\frac{1}{2}} \\ L^T & 0 & -I & 0 & 0 \\ 0 & M_1 S & 0 & -I & 0 \\ 0 & W_1^{\frac{T}{2}} S & 0 & 0 & -\varepsilon I \end{bmatrix} < 0 \quad \dots (24)$$

and the cost function satisfies the bound.

$$\text{Ɉ}_{0_1} < \sum_{l=0}^{\varrho_1 - 1} X_\theta^{h^T}(0,l)\left(\frac{P_\theta^h + P_\theta^T}{2}\right) X_\theta^h(0,l) + \sum_{k=0}^{\varrho_2 - 1} X_\theta^{v^T}(k,0)\left(\frac{P_\theta^v + P_\theta^T}{2}\right) X_\theta^v(k,0) \quad \dots (25)$$

Proof: By utilizing equations (3), Lemma 1, and (12), it is possible to reorganize the expressions as follows:

$$\begin{bmatrix} \left(\frac{P_\theta + P_\theta{}^T}{2}\right) + \varepsilon H H^T & A \\ A^T & \varepsilon^{-1} M_1^T M_1 - \left(\frac{P_\theta + P_\theta{}^T}{2}\right) + W_1 \end{bmatrix} > 0 \quad \dots (26)$$

Pre- and post-multiplying (26) by the matrix

$$\begin{bmatrix} \varepsilon^{\frac{1}{2}} I & 0 \\ 0 & \varepsilon^{\frac{1}{2}} \left(\frac{P_\theta + P_\theta{}^T}{2}\right)^{-1} \end{bmatrix} \quad \dots (27)$$

where,

$$\varepsilon = SP_e = S\left(\frac{P_\theta + P_\theta{}^T}{2}\right) \quad \dots (28)$$

The equivalence between Eq. (24) and (22) can be easily deduced from the Schur complements. Additionally, the cost function in Eq. (25) bound can be readily obtained using Eq. (24), as shown.

Remarks 1. It is important to note that the matrix inequality (24) is a linear function with respect to the variables S, U and ε.

Robust Guaranteed Cost Control Via Static State Feedback

This section aims to determine SSF (static-state feedback) $u(k,l) = KX_\theta(k,l)$. For the system Eq. (1) and cost function Eq. (6), the feedback should ensure that the closed-loop system is asymptotically stable and that the closed-loop cost function does not exceed a specific upper bound.

$$\begin{bmatrix} X_\theta^h(k,l+1) \\ X_\theta^v(k+1,l) \end{bmatrix} = (\text{A} + \Delta\text{A} + K\text{B} + K\Delta\text{B}) \begin{bmatrix} X_\theta^h(k,l) \\ X_\theta^v(k,l) \end{bmatrix} \quad \dots (29)$$

Moreover, the cost function Eq. (6) simplifies to:

$$\text{Ɉ} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} X_\theta^T(k,l) W_2 X_\theta(k,l) \quad \dots (30)$$

where,

$$W_2 = W_1 + K^T R K \quad \dots (31)$$

satisfies $\text{Ɉ} \le \text{Ɉ}^*$ where Ɉ^* is some specified constant.

$$\text{Ɉ}_{0_1} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} X_\theta^T(k,l) W_1 X_\theta(k,l) \quad \dots (31)$$

Remarks 2. A control law $u(k,l)$ is called a guaranteed cost control law for the uncertain system (2.1) if it satisfies two vital conditions:

(1) Ensuring asymptotic stability for the closed-loop system (26) across all admissible uncertainties.
(2) The closed-loop value of the cost function (27) is$\text{Ɉ} \le \text{Ɉ}^*$. The scalar value Ɉ^* is referred to as the guaranteed cost.

Definition 3: If there exists a positive definite symmetric matrix (PDSM) W_2, as defined by equation (29), such that the state feedback controller $u(k,l)$ satisfies the quadratic guaranteed cost control for the system (29) and cost function (30) associated with the cost matrix

$$[\text{A} + \Delta\text{A} + K\text{B} + K\Delta\text{B}]^T \left(\frac{P_\theta + P_\theta{}^T}{2}\right) [\text{A} + \Delta\text{A} + K\text{B} + K\Delta\text{B}] - \left(\frac{P_\theta + P_\theta{}^T}{2}\right) + W_2 > 0 \quad \dots (32)$$

for all $\|F(k,l)\| \le 1$ For every value of $F(k,l)$ such that $\|F(k,l)\|$ is less than or equal to 1.

Theorem 3. For a positive scalar ε and a matrix U of $m \times n$ dimension with zero initial conditions, there exists a matrix P_θ not necessarily symmetric such that the following LMI Holds

$$\begin{bmatrix} -S & \overline{\text{A}}_1 & L & 0 & 0 & 0 \\ \overline{\text{A}}_1^T & -S & 0 & \overline{D_1} & SW_1^{\frac{1}{2}} & U^T R^{\frac{1}{2}} \\ L^T & 0 & -I & 0 & 0 & 0 \\ 0 & \overline{D_1}^T & 0 & -I & 0 & 0 \\ 0 & W_1^{\frac{T}{2}} S & 0 & 0 & -\varepsilon I & 0 \\ 0 & R^{\frac{T}{2}} U & 0 & 0 & 0 & -\varepsilon I \end{bmatrix} < 0 \quad \dots (33)$$

then, there exists a static state controller of the form $u(k,l) = KX_\theta(k,l)$ that solves the robust cost control problem.

where,

$$S = \varepsilon P_{\theta}^{-1} = \varepsilon\left(\frac{P_{\theta} + P_{\theta}^{T}}{2}\right) = \text{diag}\,\{S^{h}, S^{v}\} > 0,$$

$$\overline{A}_1 = \overline{A\,S + BU}, \overline{D_1} = SM_1^T + U^T M_2^T.$$

In this situation, a suitable control law is given by $K = US^{-1}$ which ensures the guaranteed cost. Moreover, the cost function (21) satisfies the bound.

$$Ɉ < \varepsilon\begin{bmatrix} (\varrho_1 - 1)\lambda_{max}\left(M^T S^{h^{-1}} M\right) \\ +(\varrho_2 - 1)\lambda_{max}\left(M^T S^{v^{-1}} M\right)\end{bmatrix} \quad \text{... (34)}$$

Optimization problem

Minimize $(\varrho_1\alpha + \varrho_2\beta)$

$$s.t.\begin{cases} (33) \\ \begin{bmatrix} -\alpha I & M^T \\ M & S^{h}\end{bmatrix} \\ \begin{bmatrix} -\beta I & M^T \\ M & S^{v}\end{bmatrix}\end{cases} \quad \text{... (35)}$$

Remarks 3. The linearity of variables S, U andε in matrix inequality Eq. (33) carried out the use of the MATLAB robust control toolbox to determine if a static-state feedback controller exists and to obtain the least upper bound on the guaranteed cost. The upper bound of the closed loop cost function is evidently influenced by the selection of the guaranteed cost controllers.[19,20]

The aforementioned inequality, together with Eq. (35), leads to the conclusion that the uncertain free system is asymptotically stable under all admissible uncertainties, according to Lemma 1.

Results and Discussion

This has provided the mathematical approach to analyze the asymptotic stability of 2-D Roesser models in various circumstances.

Example 4.1 Let us now concentrate on such a two-dimensional discrete Roesser model specified by the uncertain system Eq. (1)

$$A = \begin{bmatrix} 0 & 4 & 0 \\ -0.2 & 0.7 & 0 \\ 0 & 0 & 0.01\end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1\end{bmatrix}, L = \begin{bmatrix} 1 \\ 0 \\ 0\end{bmatrix},$$

$$W = \begin{bmatrix} 0.0025 & 0.0084 & 0 \\ 0.0084 & 0.0025 & 0 \\ 0 & 0 & 0.0044\end{bmatrix}$$

$$M_1 = [0 \quad 1 \quad -1], M_2 = 0, Z = 0.1,$$

$$R = 0.20, \varrho_1 = \varrho_2 = 1.$$

The eigenvalues of matrix A determine the stability of the system. When all of A eigenvalues lie within the complex plane's unit circle, the system is asymptotically stable. If any of the eigenvalues are beyond the unit circle, it is expressed as follows[21]

$$N\left(Z_1, Z_2\right) = det\begin{bmatrix} I_m - Z_1 A_1 & -Z_1 A_2 \\ -Z_2 A_3 & I_n - Z_2 A_4\end{bmatrix},$$

$$N\left(Z_1, Z_2\right) = det\begin{bmatrix} 1 & -4Z_1 & 0 \\ 0.2Z_1 & 0.3Z_1 & 0 \\ 0 & 0 & 1 - 0.01Z_2\end{bmatrix}$$

$$= 0.3Z_1(1 - 0.1Z_2) + 4Z_1(0.2Z_1 - 0.002Z_1 Z_2)$$

Then, the state space model is asymptotically stable if and only if

$$N\left(Z_1, Z_2\right) \neq 0 \text{for all} \left(Z_1, Z_2\right) \in \overline{U}^2 \quad \text{... (36)}$$

where, $\overline{U}^2 = \{(Z_1, Z_2): |Z_1| \leq 1, |Z_2| \leq 1\}, Z$ is a complex number and $I = I_m \oplus I_n$ denote the identity matrix. It is seen that Eq. (36) violated (choose, for instance, $Z_1 = 0.94$, $Z_2 = 98/3426$), for the characteristic polynomial. The system is considered unstable if any of the eigenvalues have a magnitude exceeding 1. Conversely, if all the eigenvalues possess magnitudes that are less than or equal to 1, the system is observed as asymptotically stable.

From the outcome, the analyzed system does not exhibit 2-D asymptotic stability[20]. The objective is to create an optimal controller with a guaranteed cost for this system. For this specific scenario, this were able to determine through the use of the LMI Toolbox in MATLAB[21,22] that it is feasible to solve the optimization problem presented in Eq. (35).

Using the MATLAB robust control toolbox gives $t_{min} = -5.7076 \times 10^{-4}$ which is negative. This negative value suggests that the system is feasible. The optimal solutions for this problem are as follows:

$$S = \begin{bmatrix} 3.2829 & 0.2978 & 0 \\ 0.1514 & 0.1261 & 0 \\ 0 & 0 & 0.1172\end{bmatrix},$$

$$U = [\,1.7151 \times 10^{-8} \quad 0.0381 \quad 4.4093 \times 10^{-6}]$$

$$\varepsilon = 0.0673, \alpha = 0.0032, \beta = 0.0853$$

The controller that provides the optimal guaranteed cost for this system, as per Theorem 3, is given by $u(k, l) = [-0.015 \quad 0.3119 \quad -0.034]X_{\theta}(k, l)$

Furthermore, the least upper bound for the associated closed-loop cost function is $Ɉ^* = 5.5601 \times 10^{-5}$.

In an open-loop response, a system reacts to a control input applied directly without any feedback to adjust based on its output. This type of response in Fig. 1 an unstable system typically shows divergence, indicating increasing instability. Conversely, a closed-loop response involves feedback control in Fig. 2 where the control input is adjusted based on the system's output. For a stable system, this response shows convergence to a steady state, demonstrating stability.

Example 4.2 Let us now concentrate on such a 2-D discrete Roesser model specified by the uncertain system Eq. (1)

$$A = \begin{bmatrix} -1.01 & -0.22 & 0.50 \\ 0.60 & -4.50 & 0.01 \\ 0.70 & -2.20 & 0.10 \end{bmatrix}, B = \begin{bmatrix} 0.10 \\ 0.20 \\ 0.10 \end{bmatrix}, L = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$W = \begin{bmatrix} 0.0025 & 0.0064 & 0 \\ 0.0064 & 0.0025 & 0 \\ 0 & 0 & 0.0020 \end{bmatrix},$$

$M_1 = [0 \quad 0 \quad -1], M_2 = 0, Z = 0.1,$
$R = 0.25, \varrho_1 = \varrho_2 = 2$

Using the MATLAB robust control toolbox gives $t_{min} = -0.0163$, which is negative. Based on the current example, this has determined that the optimization problem can be feasibly solved. The optimal solutions for this problem are as follows:

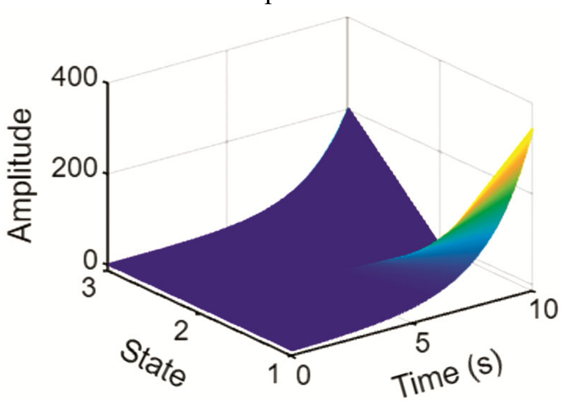

Fig. 1 — Open-loop response of a stable system

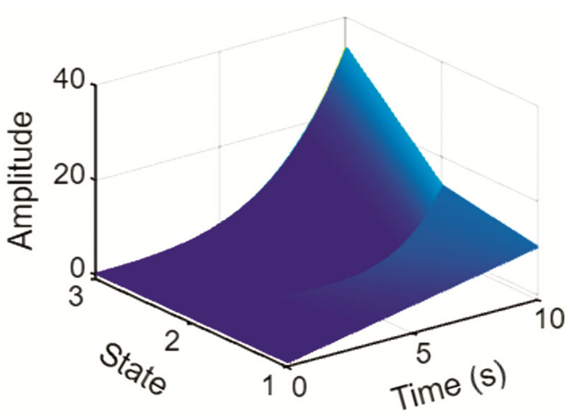

Fig. 2 — Closed-loop response of a stable system

$$S = \begin{bmatrix} 1.97 & 3.50 & 0 \\ 0.95 & 2.35 & 0 \\ 0 & 0 & 0.65 \end{bmatrix},$$

$U = [11.91 \quad 34.93 \quad -2.13]$
$\varepsilon = 338.8, \alpha = 0.0065 \beta = 0.0154$

The controller that provides the optimal guaranteed cost for this system, as per Theorem 3, is given by

$$u(k,l) = [-4.59 \quad 21.6 \quad -3.26] X_\theta(k,l)$$

Furthermore, the least upper bound for the associated closed-loop cost function is$J^* = 0.1025$

In the context of a 2-D uncertain Roesser model, an unsymmetric Lyapunov matrix can be used to establish stability conditions, which can be expressed in the form of LMIs. This feature facilitates traceable mathematical manipulation, allowing it to be solved using MATLAB robust control toolbox[21,22], as the lothisr values of the cost function indicate better performance. A numerical example has been included to illustrate the effectiveness of the current technique in order to establish its authenticity.

Conclusions

In this paper, this presented a novel LMI-based approach for designing an optimal guaranteed cost control via a static-state feedback controller for uncertain two-dimensional discrete systems using the Roesser model. By utilizing an unsymmetrical Lyapunov matrix, this has derived a new form of LMI. The paper has established LMI-based sufficient conditions for the existence of a guaranteed cost controller. An optimal GCC controller has been designed based on the solution of a convex optimization problem. This controller not only ensures the asymptotic stability of the system but also achieves an optimized level of better cost function. The results obtained from several illustrative examples demonstrated the superiority of the proposed approach using an unsymmetrical Lyapunov matrix in achieving the optimized cost function. Future work focuses on extending this approach to more complex system configurations or exploring its applicability in robust stability analysis of two-dimensional non-linear digital filters appears to represent an interesting and challenging subject.

Conflict of Interest

The authors declare that they have no conflict of interest in the publication of this research.

Acknowledgment

We sincerely thank the editor and reviewers for their insightful comments and constructive suggestions, which have significantly improved our manuscript.

References

1 Dhawan A & Kar H, An LMI approach to robust optimal guaranteed cost control of 2-D discrete systems described by the Roesser model, *Signal Process*, **90(9)** (2010) 2648–54, doi: 10.1016/j.sigpro.2010.03.008.

2 Kanellakis A, Tawfik A & Agathoklis P, Stability analysis of 2-D discrete and continuous state-space systems, *IET Control Theory A*, **16(3)** (2022) 273–82, doi.org/10.1049/cth2.12224.

3 Wang F, Wang Z, Liang J & Yang J, A survey on filtering issues for two-dimensional systems: Advances and challenges, *Int J Control Autom*, **18(3)** (2020) 629–642, doi: 10.1007/s12555-019-1000-x.

4 Wang J, Guo T & Xiang P, Stability analysis and control synthesis of uncertain Roesser-type discrete-time two-dimensional systems, *Chinese Phys B*, **22(3)** (2013) 030206, doi: 10.1088/1674-1056/22/3/030206.

5 Tandon A, Dhawan A & Tiwari M, Optimal guaranteed cost control of uncertain 2-D discrete state-delayed systems described by the Roesser model via memory state feedback, *T I Meas Control*, **41(1)** (2019) 285–94, doi:10.1177/0142331218754623.

6 Massera C M, Terra M H & Wolf D F, Optimal guaranteed cost control of discrete-time linear systems subject to structured uncertainties, *Int J Control*, **94(4)** (2021) 1132–1142, doi: 10.1080/00207179.2019.1634838.

7 Boudellioua S, Krzysztof G & Rogers E, Equivalent 2-D nonsingular Roesser models for discrete linear repetitive processes, *Int J Control*, **91(12)** (2018) 2673–268, doi: 10.1080/00207179.2017.1414307.

8 Pandiya G P, Vidyarthi A, Tiwari M, Dhawan A & Maheswar R, Optimal Guaranteed Cost Control of an Uncertain and Shift-Delayed 2-D Discrete FM First Model via Memory State Feedback, In *Advances in VLSI, Communication, and Signal Processing: Select Proceedings of VCAS* (Singapore: Springer Nature Singapore) 2021, 405–419., 2022.doi.org/10.1007/978-981-19-2631-0_36.

9 Pandey S, Tadepalli K, Leite S, Nigam R & Bhusnur S, Stability of 2-D discrete systems in the presence of saturation function and delays, *Int J Control Autom*, **21(3)** (2023) 788–799, doi: 10.1007/s12555-021-1061-5.

10 Chaurasia D, Kalpana, Singh V, Kandanvli K R & Kar H, Stability of uncertain 2-D discrete delayed systems with saturation, *Int J Adv Tech Eng Explor*, **91(9)** (2022) 771, doi: 10.19101/IJATEE.2021.875259.

11 Ettore F & Giovanni M, State-space realization theory of two-dimensional filters, *IEEE T Automat Contr*, **21(4)** (1976) 484–92, doi: 10.1109/tac.1976.1101305.

12 Ettore F & Giovanni M, Doubly-indexed dynamical systems: State-space models and structural properties, *Math Syst Theory*, **12(1)** (1978) 59–72.

13 Kurek, J, The general state-space model for a two-dimensional linear digital system, *IEEE T Automat Contr*, **30(6)** (1985) 600–602, doi: 10.1109/TAC.1985.1103998.

14 Roesser R A, Discrete state-space model for linear image processing, *IEEE T Automat Contr*, **20(1)** (1975) 1–10, doi: 10.1109/tac.1975.1100844.

15 Barro M & Traore S, An uncertain minimization problem: robust optimization versus optimization of robustness, *Optim Lett*, *(*2022) 1–20, doi: 10.1007/s11590-022-01965-8.

16 Joseph S & Vidyarthi A, Multiclass gastrointestinal diseases classification based on hybrid features and duo feature selection, *J Biomed Nanotechnol*, **19(2)** (2023) 288–298, doi.org/10.1166/jbn.2023.3533.

17 Singh V, New approach to stability of 2-D discrete systems with state saturation, *Signal Process*, **92(1)** (2012) 240–247, doi: 10.1016/j.sigpro.2011.07.012.

18 Kanellakis A & Tawfik A, Sufficient criterion for instability of 2-D linear discrete systems, *Feri OP*, **3** (2023) 100019, https://doi.org/10.1016/j.fraope.2023.100019.

19 Chesi, G. Analyzing stability in 2D systems via LMIs: From pioneering to recent contributions, *Ann Rev Control*, **57** (2024) 100933.

20 Zhu Z & Jun-Guo L, Robust stability and stabilization of continuous-discrete fractional-order 2D Fornasini–Marchesini second model with interval uncertainties, *Asian J Control*, **26** (2024) 2054–3066, https://doi.org/10.1002/ asjc.3380.

21 Boyd S, Laurent E G, Eric F & Venkataramanan B, Linear matrix inequalities in system and control theory, *SIAM Stud Appl Math*, **15** (1994).

22 Pascal G, Nemirovskii A, Laub A J & Mahmoud C, The LMI control toolbox, In *Proceedings of 1994 33rd IEEE conference on decision and control* (IEEE) 1994, 2038–2041, doi: 10.1109/CDC.1994.411440.