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A Single–Degree–of–Freedom Solution Procedure to Determine Dynamic Characteristics of Air–Bearing

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Appendix A — Governing Equation for Pressure Distribution in Air–Bearing

An order–of–magnitude analysis[15,16] yields the approximated governing equations in the cylindrical coordinates as:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \mathbf{u}) = 0 \quad \text{... (A1)}$$

$$0 \cong \frac{\partial p}{\partial r} \quad \text{... (A2)}$$

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z}\right) \cong -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[\mu r^2\left(\frac{\partial u_\theta}{\partial r}\right)\right] \quad \text{... (A3)}$$

$$0 \cong -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left[\mu r\left(\frac{\partial u_z}{\partial r}\right)\right] \quad \text{... (A4)}$$

$$\rho\left[\frac{\partial (c_v T)}{\partial t} + u_r\frac{\partial (c_v T)}{\partial r} + \frac{u_\theta}{r}\frac{\partial (c_v T)}{\partial \theta} + u_z\frac{\partial (c_v T)}{\partial z}\right] \cong k\left[\frac{\partial^2 T}{\partial r^2}\right] \quad \text{... (A5)}$$

where, ρ is fluid density; $\mathbf{u}$, velocity; μ, dynamic viscosity; T, temperature; c_v, constant–volume specific heat; k, thermal conductivity; u_r, radial component of $\mathbf{u}$; u_θ, circumferential component of $\mathbf{u}$; and u_z, axial component of $\mathbf{u}$. p does not vary with r as shown by Eq. (A2). The order–of–magnitude of the terms on the Left–Hand Side (LHS) of Eqs (A3) and (A5) are smaller than the ones on the Right–Hand Side (RHS) of the respective equations. Based on this reason, by neglecting the terms that represent the inertia forces on the LHS of Eq. (A3) and assuming an isothermal operating condition of AB, one may arrive at the RE to analyze the fluid flow through AB.[16]

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[\mu r^2\left(\frac{\partial u_\theta}{\partial r}\right)\right] \cong 0 \quad \text{... (A6)}$$

$$-\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left[\mu r\left(\frac{\partial u_z}{\partial r}\right)\right] \cong 0 \quad \text{... (A7)}$$

$$k\left[\frac{\partial^2 T}{\partial r^2}\right] \cong 0 \quad \text{... (A8)}$$

The RE is arrived at using the approximated form of the momentum and energy Eqs (A2), (A6)–(A8) as follows. The equation governing the air flow through the CV (see Fig. 4) can be written using Eq. (A1) as

$$\iiint_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right] dV = 0 \qquad \text{... (A9)}$$

Due to the motion of the journal and the bushing, the size of the CV (see Fig. 4) in δ direction is variable. Writing Eq. (A9) for a unit area of CV in θ–z plane, in terms of the dimensions $R_J d\theta$, $d\delta$, and dz and velocities u_θ, u_δ, and u_z,

$$\int_0^{h_{AB}} \left[\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u_\theta)}{R_J\partial\theta} + \frac{\partial(\rho u_\delta)}{\partial\delta} + \frac{\partial(\rho u_z)}{\partial z}\right] d\delta = 0 \qquad \text{... (A10)}$$

Using the Leibnitz rule for the integration of a differential, the four terms in Eq. (A10) are written as follows:

The fluid film is assumed to operate in isothermal conditions[28] to arrive at the RE. Considering air as an ideal gas, $\rho = \rho(p)$ only. Since $p \neq p(\delta)$ from Eq. (A2), $\partial\rho/\partial\delta = 0$.

$$\int_0^{h_{AB}} \frac{\partial\rho}{\partial t} d\delta = \frac{\partial\rho}{\partial t}\int_0^{h_{AB}} d\delta = h_{AB}\frac{\partial\rho}{\partial t} \qquad \text{... (A11)}$$

$$\begin{aligned}\int_0^{h_{AB}} \frac{\partial(\rho u_\theta)}{R_J\partial\theta} d\delta &= \frac{\partial}{R_J\partial\theta}\left(\rho\int_0^{h_{AB}} u_\theta\, d\delta\right) - \rho\left(u_\theta|_{h_{AB}}\right)\frac{\partial h_{AB}}{R_J\partial\theta} \\ &= \frac{\partial}{R_J\partial\theta}\left(\rho\int_0^{h_{AB}} u_\theta\, d\delta\right) - \rho R_J\omega\frac{\partial h_{AB}}{R_J\partial\theta}\end{aligned} \qquad \text{... (A12)}$$

As the bushing does not rotate, $u_\theta|_0 = 0$ in Eq. (A12).

$$\int_0^{h_{AB}} \frac{\partial(\rho u_\delta)}{\partial\delta} d\delta = \rho[u_\delta]_0^{h_{AB}} = \rho\left(u_\delta|_{h_{AB}} - u_\delta|_0\right) = \rho\frac{dh_{AB}}{dt} \qquad \text{... (A13)}$$

$$\int_0^{h_{AB}} \frac{\partial(\rho u_z)}{\partial z} d\delta = \frac{\partial}{\partial z}\left(\rho\int_0^{h_{AB}} u_z\, d\delta\right) \qquad \text{... (A14)}$$

As the bushing does not have an axial motion, $u_z|_{h_{AB}} = u_z|_0 = 0$ in Eq. (A14). Since $h_{AB}=h_{AB}(\theta, t)$,

$$\frac{dh_{AB}}{dt} = \frac{\partial h_{AB}}{R_J\partial\theta}\frac{R_J d\theta}{dt} + \frac{\partial h_{AB}}{\partial t} \qquad \text{... (A15)}$$

where, $d\theta/dt = \omega$. Using Eq. (A15) in Eq. (A13),

$$\int_0^{h_{AB}} \frac{\partial(\rho u_\delta)}{\partial\delta} d\delta = \rho R_J\omega\frac{\partial h_{AB}}{R_J\partial\theta} + \rho\frac{\partial h_{AB}}{\partial t} \qquad \text{... (A16)}$$

The RE is derived by assuming μ as a constant. u_θ and u_z in Eqs (A12) and (A14) are given by

$$u_\theta = \frac{1}{2\mu\left(R_J + c_{AB}\right)}\frac{\partial p}{\partial\theta}\delta^2 + C_1\delta + C_2 \qquad \text{... (A17)}$$

$$u_z = \frac{1}{2\mu}\frac{\partial p}{\partial z}\delta^2 + C_3\delta + C_4 \qquad \text{... (A18)}$$

Equations (A17) and (A18) are arrived at from Eqs (A6) and (A7), respectively, as follows: (i) writing $r = R_J + c_{AB} - \delta$ in Eqs (A6) and (A7), (ii) eliminating terms of $\delta / (R_J + c_{AB})$ as they are of $O(c_{AB}/(\pi D_J))$, and (iii) integrating the resulting equations twice.

When the conditions on the journal and the bushing boundaries, $u_\theta|_0 = 0, u_\theta|_{h_{AB}} = R_J\omega,$ and $u_z|_0 = u_z|_{h_{AB}} = 0$, are applied, the velocity distributions applicable for the AB are:

$$u_\theta = \frac{1}{2\mu R_J}\frac{\partial p}{\partial \theta}(\delta^2 - h_{AB}\delta) + R_J\omega\left(\frac{\delta}{h_{AB}}\right) \quad \text{... (A19)}$$

where, $R_J + c_{AB} \approx R_J$.

$$u_z = \frac{1}{2\mu}\frac{\partial p}{\partial z}(\delta^2 - h_{AB}\delta) \quad \text{... (A20)}$$

Substituting for u_θ and u_z in Eqs (A12) and (A14) from Eqs (A19) and (A20) respectively,

$$\int_0^{h_{AB}} \frac{\partial(\rho u_\theta)}{R_J\partial\theta}d\delta = \frac{\partial}{R_J\partial\theta}\left(-\frac{\rho h_{AB}^3}{12\mu}\frac{\partial p}{R_J\partial\theta} + \frac{\rho R_J\omega h_{AB}}{2}\right) - \rho R_J\omega\frac{\partial h_{AB}}{R_J\partial\theta} \quad \text{... (A21)}$$

$$\int_0^{h_{AB}} \frac{\partial(\rho u_z)}{\partial z}d\delta = \frac{\partial}{\partial z}\left(-\frac{\rho h_{AB}^3}{12\mu}\frac{\partial p}{\partial z}\right) \quad \text{... (A22)}$$

Substituting for the four terms in Eq. (A10) from Eqs (A11), (A16), (A21), and (A22),

$$h_{AB}\frac{\partial \rho}{\partial t} - \frac{\partial}{R_J\partial\theta}\left(\frac{\rho h_{AB}^3}{12\mu R_J}\frac{\partial p}{\partial\theta}\right) + \frac{R_J\omega}{2}\frac{\partial(\rho h_{AB})}{R_J\partial\theta} + \rho\frac{\partial h_{AB}}{\partial t} - \frac{\partial}{\partial z}\left(\frac{\rho h_{AB}^3}{12\mu}\frac{\partial p}{\partial z}\right) = 0 \quad \text{... (A23)}$$

The terms on the LHS of Eq. (A23) represent the time rate of change of mass inside the finite CV having unit area in the θ–z plane, due to the following physical mechanisms: The first term represents the local rate of change in density. The second term and the fifth term are due to the Poiseuille flow (p–induced flow) mechanism. The third term is due to the Couette flow (velocity–induced flow) mechanism. *Since the bushing doesn't rotate, this term would not be present in the RE to compute the p–distribution in the AR region of ARB. But the identical RE were used for calculating p–distribution in AR and AB regions in the 1–DOF solution procedure.*[7] The fourth term in Eq. (A23) is due to the squeeze film mechanism. Equation (A23) is known as the RE. p distribution is computed using this equation in locations other than the FH in the AB. In the neighborhoods of FH, the time rate of change of mass of the finite CV is equal to the mass flow entering the AB region through the FH. At the FH locations,

$$\frac{\partial(\rho h_{AB})}{\partial t} - \frac{\partial}{R_J\partial\theta}\left(\frac{\rho h_{AB}^3}{12\mu R_J}\frac{\partial p}{\partial\theta}\right) + \frac{R_J\omega}{2}\frac{\partial(\rho h_{AB})}{R_J\partial\theta} - \frac{\partial}{\partial z}\left(\frac{\rho h_{AB}^3}{12\mu}\frac{\partial p}{\partial z}\right) = \frac{d\dot{m}_{FH}}{R_J d\theta dz} \quad \text{... (A24)}$$

Since $\rho = \rho(p)$ and $p = \rho \mathrm{R}T$ (the ideal gas equation of state where, R is the gas constant for air), Eq. (A24) becomes Eq. (6).

Appendix B — Governing Equations for Mass Flow Through the Feed–Holes of Air–Bearing

The following conditions are assumed to exist for a finite CV (see Fig. 4) having unit area in the *θ*–*z* plane, at the location of an FH in the AB region: (a) a viscous and steady flow; (b) the time rate of change of mass inside the CV due to the velocity induced flow mechanism and the squeeze film mechanism are negligible; (c) the variations of h_{AB} in θ and z directions are negligible.[7] By applying these conditions in Eq. (8), the following Poisson's equation is obtained that governs the flow at the FH.

$$-\left(H_{i,j}^{\mathrm{FH}}\right)^3\left(\frac{\partial^2 Q_{i,j}^{\mathrm{FH}}}{\partial\theta^2}+\frac{\partial^2 Q_{i,j}^{\mathrm{FH}}}{\partial\xi^2}\right)=2\frac{\dot{m}_{\mathrm{FH}}}{\mathrm{C}_1 d\theta d\xi} \quad \text{... (B1)}$$

Replacing the partial derivatives in Eq. (B1) with second–order, central differences, using $\mathrm{C}_2 = \Delta\theta/\Delta\xi$, and re–arranging,

$$Q_{i,j}^{\mathrm{FH}}=\frac{1}{2\left[\mathrm{C}_2+\frac{1}{\mathrm{C}_2}\right]}\left[\left(Q_{i+1,j}^{\mathrm{FH}}+Q_{i-1,j}^{\mathrm{FH}}\right)\mathrm{C}_2+\frac{\left(Q_{i,j+1}^{\mathrm{FH}}+Q_{i,j-1}^{\mathrm{FH}}\right)}{\mathrm{C}_2}+\frac{2\dot{m}_{\mathrm{FH}}}{\mathrm{C}_1\left(H_{i,j}^{\mathrm{FH}}\right)^3}\right] \quad \text{... (B2)}$$

Equation Governing Mass Flow Through Annular Orifice of Feed–Holes

Flow occurs from the FH chamber (where, $p = p_{\mathrm{c}}$) into AB (see Fig. 6) through the area of annular orifice (ao), $A_{\mathrm{ao}} = \pi d_{\mathrm{c}} h_{i,j}^{\mathrm{FH}}$. In the RHS of Eq. (B2), $\dot{m}_{\mathrm{FH}} = \upsilon_{\mathrm{FH}}\dot{m}_{\mathrm{FH,cr}}$, where, υ_{FH} is dimensionless mass flow rate; and $\dot{m}_{\mathrm{FH,cr}}$, critical mass flow rate through A_{ao}. The theoretical value of $\dot{m}_{\mathrm{FH,cr}}$ for compressible flow based on the assumptions: (a) frictionless flow, (b) no heat transfer, (c) negligible body forces, and (d) one–dimensional flow, is

$$\dot{m}_{\mathrm{FH,cr}}=p_{\mathrm{c}}\left(\pi d_{\mathrm{c}} h_{i,j}^{\mathrm{FH}}\right)\beta^{(\gamma+1)/2\gamma}\sqrt{\frac{\gamma}{\mathrm{R}T}} \quad \text{for} \quad \frac{p_{i,j}^{\mathrm{FH}}}{p_{\mathrm{c}}}=\beta \quad \text{... (B3)}$$

where, γ is specific heat ratio; and β, critical pressure ratio.[30]

Defining dimensionless FH chamber pressure ratio $\pi_{\mathrm{FH}} = p_{\mathrm{c}}/p_{\mathrm{s}}$, and substituting for $\dot{m}_{\mathrm{FH,cr}}$ from Eq. (B3), Eq. (B2) becomes

$$Q_{i,j}^{\mathrm{FH}}=\frac{1}{2\left[\mathrm{C}_2+\frac{1}{\mathrm{C}_2}\right]}\left[\left(Q_{i+1,j}^{\mathrm{FH}}+Q_{i-1,j}^{\mathrm{FH}}\right)\mathrm{C}_2+\frac{\left(Q_{i,j+1}^{\mathrm{FH}}+Q_{i,j-1}^{\mathrm{FH}}\right)}{\mathrm{C}_2}+\frac{\mathrm{C}_3\upsilon_{\mathrm{FH}}\pi_{\mathrm{FH}}}{\mathrm{C}_1\left(H_{i,j}^{\mathrm{FH}}\right)^2}\right] \quad \text{... (B4)}$$

where, C_3 is given by

$$\mathrm{C}_3 = 2\pi p_{\mathrm{s}} d_{\mathrm{c}} c_{\mathrm{AB}}\beta^{(\gamma+1)/2\gamma}\sqrt{\frac{\gamma}{\mathrm{R}T}}$$

and

$$\dot{m}_{\mathrm{FH}} = \mathrm{C}_3\upsilon_{\mathrm{FH}}\pi_{\mathrm{FH}}H_{i,j}^{\mathrm{FH}}/2 \quad \text{... (B5)}$$

Defining dimensionless pressure ratios $\pi_{\mathrm{e}} = p_{i,j}^{\mathrm{FH}}/p_{\mathrm{c}}$ and $\pi_{\mathrm{s}} = p_{\mathrm{s}}/p_{\mathrm{a}}$, Eq. (B4) can be written as

$$\left(\pi_{\mathrm{e}}\pi_{\mathrm{FH}}\pi_{\mathrm{s}}\right)^2=\frac{1}{2\left[\mathrm{C}_2+\frac{1}{\mathrm{C}_2}\right]}\left[\left(Q_{i+1,j}^{\mathrm{FH}}+Q_{i-1,j}^{\mathrm{FH}}\right)\mathrm{C}_2+\frac{\left(Q_{i,j+1}^{\mathrm{FH}}+Q_{i,j-1}^{\mathrm{FH}}\right)}{\mathrm{C}_2}+\frac{\mathrm{C}_3\upsilon_{\mathrm{FH}}\pi_{\mathrm{FH}}}{\mathrm{C}_1\left(H_{i,j}^{\mathrm{FH}}\right)^2}\right] \quad \text{... (B6)}$$

υ_{FH} is related to $\pi_{\mathrm{t}} = p_{i,j}^{\mathrm{FH,t}}/p_{\mathrm{c}}$ and β using[7,31]

$$\frac{(\pi_{\mathrm{t}} - \beta)^2}{(1-\beta)^2} + \upsilon_{\mathrm{FH}}^2 = 1 \quad \text{... (B7)}$$

The theoretical pressure drop $\left(p_{\mathrm{c}} - p_{i,j}^{\mathrm{FH,t}}\right)$ is related to the effective pressure drop $\left(p_{\mathrm{c}} - p_{i,j}^{\mathrm{FH}}\right)$ using

$$1 - \pi_{\mathrm{e}} = C_{\mathrm{FH}}(1 - \pi_{\mathrm{t}}) \quad \text{... (B8)}$$

where, C_{FH} is an experimentally determined coefficient[7,32]: (i) For Reynolds number $Re \leq 2000$, $C_{\mathrm{FH}} = 0.16 + 0.0002\,Re$; (ii) For $2000 < Re < 4000$, $C_{\mathrm{FH}} = 0.685 + 0.155\,((Re - 3000)/2000) - 0.19\,((Re - 3000)/2000)^2$; and (iii) For $Re \geq 4000$, $C_{\mathrm{FH}} = 0.715$. C_{FH} is introduced in Eq. (B8) so that the effects of (a) friction losses and (b) a sudden change in the direction of flow as well as a sudden change in cross–sectional area of the flow from the FH chamber to AB (see Fig. 6) are included.

From Eqs (B7) and (B8),

$$\pi_{\mathrm{e}} = 1 - \left[C_{\mathrm{FH}}(1-\beta)\left(1 - \sqrt{1 - \upsilon_{\mathrm{FH}}^2}\right)\right] \quad \text{... (B9)}$$

Substituting for π_{e} in Eq. (B6), Eq. (10) is arrived at.

Re relation is derived as follows: From the literature[30]

$$Re = \dot{m}_{i,j}^{\mathrm{FH}} h_{i,j}^{\mathrm{FH}} / (A_{\mathrm{ao}}\mu) \quad \text{... (B10)}$$

Substituting for $\dot{m}_{\mathrm{FH}} = \upsilon_{\mathrm{FH}} \dot{m}_{\mathrm{FH,cr}}$ in Eq. (B10),

$$Re = \upsilon_{\mathrm{FH}} \dot{m}_{\mathrm{FH,cr}} h_{i,j}^{\mathrm{FH}} / (A_{\mathrm{ao}}\mu) \quad \text{... (B11)}$$

Substituting for $\dot{m}_{\mathrm{FH,cr}}$ from Eq. (B3),

$$Re = \frac{\upsilon_{\mathrm{FH}} p_{\mathrm{c}} \beta^{(\gamma+1)/2\gamma} h_{i,j}^{\mathrm{FH}}}{\mu} \sqrt{\frac{\gamma}{\mathrm{R}T}} \quad \text{... (B12)}$$

Using the relations for (i) $\pi_{\mathrm{FH}} = p_{\mathrm{c}}/p_{\mathrm{s}}$, and (ii) $H_{i,j}^{\mathrm{FH}} = h_{i,j}^{\mathrm{FH}}/c_{\mathrm{AB}}$

$$Re = \frac{\upsilon_{\mathrm{FH}} \pi_{\mathrm{FH}} c_{\mathrm{AB}} H_{i,j}^{\mathrm{FH}} p_{\mathrm{s}}}{\mu} \beta^{(\gamma+1)/2\gamma} \sqrt{\frac{\gamma}{\mathrm{R}T}} \quad \text{... (B13)}$$

Equation Governing Mass Flow Through Orifice of Feed–Holes

The mass flow through FH orifice (o) when $\beta < \pi_{\mathrm{FH}} < 1$[30] is

$$\dot{m}_{\mathrm{o}} = C_{\mathrm{o}} \frac{p_{\mathrm{s}} A_{\mathrm{o}}}{\sqrt{\mathrm{R}T}} {\pi_{\mathrm{FH}}}^{1/\gamma} \sqrt{\frac{2\gamma}{\gamma - 1}\left(1 - {\pi_{\mathrm{FH}}}^{(\gamma-1)/\gamma}\right)} \quad \text{... (B14)}$$

The mass flow through the FH orifice when $\pi_{\mathrm{FH}} \leq \beta$[30] is

$$\dot{m}_{\mathrm{o}} = C_{\mathrm{o}} \frac{p_{\mathrm{s}} A_{\mathrm{o}}}{\sqrt{\mathrm{R}T}} \beta^{1/\gamma} \sqrt{\frac{2\gamma}{\gamma - 1}\left(1 - \beta^{(\gamma-1)/\gamma}\right)} \quad \text{... (B15)}$$

where, $C_{\mathrm{o}} = 0.85 - 0.15\,\pi_{\mathrm{FH}} - 0.1\,\pi_{\mathrm{FH}}^2$; and $A_{\mathrm{o}} = \pi d_{\mathrm{o}}^2/4$. C_{o} is an experimentally determined discharge coefficient.[7] It is introduced in Eqs (B14) and (B15) to include the effects of friction losses and a sudden change in the cross–sectional area of the flow at the FH orifice (see Fig. 6). A dimensionless mass flow through FH orifice υ_{o} when $\beta < \pi_{\mathrm{FH}} < 1$, is defined as

$$\upsilon_{\mathrm{o}} = \frac{{\pi_{\mathrm{FH}}}^{1/\gamma}\sqrt{(1-{\pi_{\mathrm{FH}}}^{(\gamma-1)/\gamma})}}{\beta^{1/\gamma}\sqrt{(1-\beta^{(\gamma-1)/\gamma})}} \quad \text{... (B16)}$$

When $\pi_{\mathrm{FH}} \leq \beta$, $\upsilon_{\mathrm{o}} = 1$.

Matching of Mass Flow Through Annular Orifice and Orifice of Feed–Holes

The SHM of the journal during the TSS changes the mass of air m_{c} inside the FH chamber. The time rate change of m_{c} is given by

$$\lim_{\Delta t\to 0}\frac{\Delta m_{\mathrm{c}}}{\Delta t/2} = \lim_{\Delta t\to 0}\frac{p_{\mathrm{c}}-p_{\mathrm{c,0}}}{\Delta t/2}\frac{V_{\mathrm{c}}}{\mathrm{R}T} = \dot{m}_{\mathrm{o}} - \dot{m}_{\mathrm{FH}} \quad \text{... (B17)}$$

where, $V_{\mathrm{c}} = \pi d_{\mathrm{c}}^2 h_{\mathrm{c}}/4$, is volume of FH chamber; and $p_{\mathrm{c,0}}$, initial p of the FH chamber. $\Delta t = 2\pi/(\nu N)$, where N is the number of time–steps in one period of the SHM of the journal. In a TSS, a single time–step, Δt is discretized into three time–levels, n, n+1, and n+2. The time durations from time–level n to time–level n+1 and from level n+1 to level n+2 are $\Delta t/2$ each.

Substituting for $\dot{m}_{\mathrm{FH}}$ using Eq. (B3), $\dot{m}_{\mathrm{o}}$ from Eqs (B14) and (B16) in Eq. (B17)

$$\lim_{\Delta t\to 0}\frac{p_{\mathrm{c}}-p_{\mathrm{c,0}}}{\Delta t/2}\frac{V_{\mathrm{c}}}{\mathrm{R}T} = (C_{\mathrm{o}}\upsilon_{\mathrm{o}}p_{\mathrm{s}}A_{\mathrm{o}} - \upsilon_{\mathrm{FH}}p_{\mathrm{c}}A_{\mathrm{ao}})\mathrm{C}_4 \quad \text{... (B18)}$$

where,

$$\mathrm{C}_4 = \beta^{1/\gamma}\sqrt{\frac{2\gamma}{(\gamma-1)\mathrm{R}T}(1-\beta^{(\gamma-1)/\gamma})} \quad \text{... (B19)}$$

Non–dimensionalizing p in Eq. (B18) using p_{s}, Eq. (11) is arrived at.

Appendix C — Newton–Raphson Method of Finding υ_{FH} and π_{FH}

On the grid points where the FH are located, Eqs (10) and (11) are solved for mass flow parameters υ_{FH} and π_{FH}. The solution is carried out using a multivariable Newton–Raphson technique[33] as follows:

For a set of N equations in N variables, $x_1, x_2, \ldots, x_N$, where the i^{th} equation in the set is $f_i(\mathbf{x}) = 0$; $i = 1, 2, \ldots, N$, an expression for $f_i[\mathbf{x}^{(b+1)}]$ using a multivariable Taylor series expansion about $\mathbf{x}^{(b)}$ can be written as (where, b is the number of iterations):

$$f_i\left(\mathbf{x}^{(b+1)}\right) = f_i\left(\mathbf{x}^{(b)}\right) + \sum_{j=1}^{N}\left(\frac{\partial f_i}{\partial x_j}\right)_{x^{(b)}}\left(x_j^{(b+1)} - x_j^{(b)}\right) + \cdots \quad \text{... (C1)}$$

$i = 1,2,\ldots,N$.

In order for $\mathbf{x}^{(b+1)}$ to be a better estimate of the solution x than is $\mathbf{x}^{(b)}$, $f_i(\mathbf{x}^{(b+1)}) = 0$, i.e., $-\mathbf{f}^{(b)} = \mathbf{A}^{(b)}(\mathbf{x}^{(b+1)} - \mathbf{x}^{(b)})$ where, $\mathbf{A}^{(b)}$ is the Jacobian matrix evaluated at $\mathbf{x}^{(b)}$, hence it is a matrix of constants. Pre–multiplying by $\mathbf{A}^{(b)\,-1}$ on both sides, $\mathbf{x}^{(b+1)} = \mathbf{x}^{(b)} - \mathbf{A}^{(b)\,-1}\mathbf{f}^{(b)}$. It can be written in an expanded form as

$$\begin{Bmatrix} x_1^{(b+1)} \\ x_2^{(b+1)} \\ \cdot \\ \cdot \\ x_N^{(b+1)} \end{Bmatrix} = \begin{Bmatrix} x_1^{(b)} \\ x_2^{(b)} \\ \cdot \\ \cdot \\ x_N^{(b)} \end{Bmatrix} - \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdot & \cdot & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdot & \cdot & \frac{\partial f_2}{\partial x_N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_N}{\partial x_1} & \frac{\partial f_N}{\partial x_2} & \cdot & \cdot & \frac{\partial f_N}{\partial x_N} \end{bmatrix}^{-1} \begin{Bmatrix} f_1\left[x^{(b)}\right] \\ f_2\left[x^{(b)}\right] \\ \cdot \\ \cdot \\ f_N\left[x^{(b)}\right] \end{Bmatrix} \quad \text{... (C2)}$$

The use of the Newton–Raphson algorithm for several variables involves obtaining the inverse of the Jacobian matrix once in every iteration.

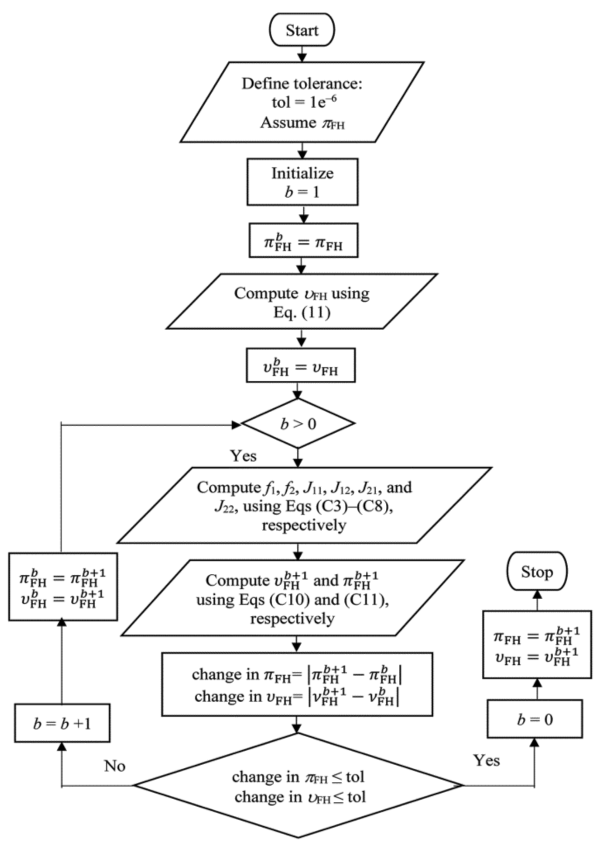
Equations (10) and (11) consist of the variables $x_1 = \upsilon_{\mathrm{FH}}$ and $x_2 = \pi_{\mathrm{FH}}$. Application of the Newton–Raphson method in these equations results in

$$f_1(b) = \left[1 - C_{\mathrm{FH}}(1-\beta)\left(1 - \sqrt{1-\upsilon_{\mathrm{FH}}^2}\right)\right]^2 (\pi_{\mathrm{FH}}\pi_{\mathrm{s}})^2 - \frac{1}{2\left(\mathrm{C}_2 + \frac{1}{\mathrm{C}_2}\right)}\left[\left(Q_{i+1,j}^{\mathrm{FH}} + Q_{i-1,j}^{\mathrm{FH}}\right)\mathrm{C}_2 + \frac{\left(Q_{i,j+1}^{\mathrm{FH}} + Q_{i,j-1}^{\mathrm{FH}}\right)}{\mathrm{C}_2} + \frac{\mathrm{C}_3\upsilon_{\mathrm{FH}}\pi_{\mathrm{FH}}}{\mathrm{C}_1\left(H_{i,j}^{\mathrm{FH}}\right)^2}\right] \quad \text{... (C3)}$$

$$f_2(b) = (C_{\mathrm{o}}\upsilon_{\mathrm{o}}A_{\mathrm{o}} - \upsilon_{\mathrm{FH}}\pi_{\mathrm{FH}}A_{\mathrm{ao}})\mathrm{C}_4 - \left(\pi_{\mathrm{FH}} - \pi_{\mathrm{FH},0}\right)\frac{V_{\mathrm{c}}}{\Delta t}\frac{1}{\mathrm{R}T} \quad \text{... (C4)}$$

$$J_{11} = \frac{\partial f_1(b)}{\partial \upsilon_{\mathrm{FH}}} = -\frac{2(\pi_{\mathrm{FH}}\pi_{\mathrm{s}})^2\upsilon_{\mathrm{FH}}C_{\mathrm{FH}}(1-\beta)\left[1 - C_{\mathrm{FH}}(1-\beta)\left(1-\sqrt{1-\upsilon_{\mathrm{FH}}^2}\right)\right]}{(1-\upsilon_{\mathrm{FH}}^2)^{\frac{1}{2}}} - \frac{\mathrm{C}_3\pi_{\mathrm{FH}}}{2\mathrm{C}_1\left(\mathrm{C}_2 + \frac{1}{\mathrm{C}_2}\right)\left(H_{i,j}^{\mathrm{FH}}\right)^2} \quad \text{... (C5)}$$

Flowchart 2 — Newton–Raphson method for solving mass flow through feed–holes



$$J_{12} = \frac{\partial f_1(b)}{\partial \pi_{\mathrm{FH}}} = 2\left[1 - C_{\mathrm{FH}}(1-\beta)\left(1 - \sqrt{1 - \upsilon_{\mathrm{FH}}^2}\right)\right]^2 \pi_{\mathrm{FH}}\pi_s^2 - \frac{\mathrm{C}_3\upsilon_{\mathrm{FH}}}{2\mathrm{C}_1\left(\mathrm{C}_2 + \frac{1}{\mathrm{C}_2}\right)\left(H_{i,j}^{\mathrm{FH}}\right)^2} \quad \text{... (C6)}$$

$$J_{21} = \frac{\partial f_2(b)}{\partial \upsilon_{\mathrm{FH}}} = -\pi_{\mathrm{FH}} A_{\mathrm{ao}} \mathrm{C}_4 \quad \text{... (C7)}$$

$$J_{22} = \frac{\partial f_2(b)}{\partial \pi_{\mathrm{FH}}} = -\upsilon_{\mathrm{FH}} A_{\mathrm{ao}} \mathrm{C}_4 - \frac{V_{\mathrm{c}}}{\Delta t}\frac{1}{\mathrm{R}T} \quad \text{... (C8)}$$

Substituting for variables x_1, x_2, functions f_1, f_2, and the elements of the Jacobian matrix from Eqs (C3)–(C8) into Eq. (C2)

$$\begin{Bmatrix} \upsilon_{\mathrm{FH}}^{b+1} \\ \pi_{\mathrm{FH}}^{b+1} \end{Bmatrix} = \begin{Bmatrix} \upsilon_{\mathrm{FH}}^{b} \\ \pi_{\mathrm{FH}}^{b} \end{Bmatrix} - \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}^{-1} \begin{Bmatrix} f_1(b) \\ f_2(b) \end{Bmatrix} \quad \text{... (C9)}$$

Solving Eq. (C9),

$$\upsilon_{\mathrm{FH}}^{b+1} = \upsilon_{\mathrm{FH}}^{b} - \left[\frac{J_{22}f_1(b) - J_{12}f_2(b)}{J_{11}J_{22} - J_{12}J_{21}}\right] \quad \text{... (C10)}$$

$$\pi_{\mathrm{FH}}^{b+1} = \pi_{\mathrm{FH}}^{b} - \left[\frac{-J_{21}f_1(b) + J_{11}f_2(b)}{J_{11}J_{22} - J_{12}J_{21}}\right] \quad \text{... (C11)}$$

In Eqs (C10) and (C11), f_1, f_2, J_{11}, J_{12}, J_{21}, and J_{22}, are given by Eqs (C3)–(C8) respectively. The computation of mass flow is carried out using the algorithm shown in Flowchart 2.

The initial value for π_{FH} (time $t = 0$ and $b = 1$) is assigned from the initial p distribution condition obtained from SSS. Using the value of π_{FH}, the value of υ_{FH} is obtained from Eq. (11). f_1, f_2, J_{11}, J_{12}, J_{21}, and J_{22} at $b = 1$ are found using Eqs (C3)–(C8), respectively. Values of υ_{FH} and π_{FH} at $(b + 1)^{\mathrm{th}}$ iteration are obtained from Eqs (C10) and (C11), respectively. The iterations continue as long as the algebraic difference between b^{th} and $(b + 1)^{\mathrm{th}}$ values of υ_{FH} and π_{FH} becomes less than a specified tolerance.

The final values of υ_{FH} and π_{FH} are implemented in Eqs (16) and (18) to compute p at the grid points that correspond to the locations of FH. During the computation of p distribution in an n^{th} time–step, the initial iteration value (at $b = 1$) of π_{FH} is taken to be the value of π_{FH} obtained in $(n–1)^{\mathrm{th}}$ time–step. The initial iteration value υ_{FH} for each time–step is obtained from Eq. (11) using the value of π_{FH}.

Appendix D — Symbolic Thomas algorithm

Equation (17) forms a singly bordered tridiagonal linear system which can be solved by using a symbolic Thomas algorithm.[35] In a symbolic Thomas algorithm, [**A**] in Eq. (17) is represented as a product of two upper triangular matrices [$\mathbf{U_1}$(λ)], [$\mathbf{U_2}$(λ)], and a lower triangular matrix [**L**(λ)].

$$[\mathbf{A}] = [\mathbf{U_1}(\lambda)]\,[\mathbf{U_2}(\lambda)]\,[\mathbf{L}(\lambda)] \quad \text{…(D1)}$$

where,

$$[\mathbf{U_1}(\lambda)] = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & f_{i,3} & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & f_{i,N-2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & f_{i,N-1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & f_{i,N} & 1 \end{bmatrix};$$

$$[\mathbf{U_2}(\lambda)] = \begin{bmatrix} h_{i,1} & c_{i,2} & 0 & 0 & \dots & 0 & 0 & 0 & \alpha_{i,1} \\ 0 & h_{i,2} & c_{i,3} & 0 & \dots & 0 & 0 & 0 & \alpha_{i,2} \\ 0 & 0 & h_{i,3} & c_{i,4} & \dots & 0 & 0 & 0 & \alpha_{i,3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & h_{i,N-2} & c_{i,N-1} & \alpha_{i,N-2} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & h_{i,N-1} & \alpha_{i,N-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & h_{i,N} \end{bmatrix};$$

$$[\mathbf{L}(\lambda)] = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \beta_{i,2} & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \beta_{i,3} & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \beta_{i,N-2} & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ \beta_{i,N-1} & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ \beta_{i,N} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix};$$

and λ, a symbolic variable. λ is substituted for $h_{i,j}$ when $h_{i,j} = 0$. From Eqs (D1) and (17), the relations between the elements of [**A**], [$\mathbf{U_1}$(λ)], [$\mathbf{U_2}$(λ)], and [**L**(λ)] are

$$\alpha_{\mathrm{i},1} = a_{i,N} \quad \text{... (D2)}$$

$$\alpha_{i,j} = 0 \qquad 2 \le j \le N-2 \quad \text{... (D3)}$$

$$\alpha_{i,N-1} = c_{i,N} \quad \text{... (D4)}$$

$$h_{i,2} = b_{i,2} \quad \text{... (D5)}$$

$$f_{i,j} = a_{i,j-1}/h_{i,j-1} \qquad 3 \le j \le N \quad \text{... (D6)}$$

$$h_{i,j} = b_{i,j} - f_{i,j}c_i \qquad 3 \le j \le N \quad \text{... (D7)}$$

$$\beta_{i,N} = \frac{1}{h_{i,N}}\left(c_{i,1} + f_{i,N}f_{i,N-1}f_{i,N-2} \cdots f_{i,5}f_{i,4}f_{i,3}a_{i,1}\right) \quad \text{... (D8)}$$

$$\beta'_{i,N} = f_{i,N}f_{i,N-1}f_{i,N-2} \cdots f_{i,5}f_{i,4}f_{i,3} \quad \text{... (D9)}$$

$$\beta'_{i,j} = \frac{\beta'_{i,j+1}}{f_{i,j+1}} \qquad N-1 \ge j \ge 3 \quad \text{... (D10)}$$

$$\beta_{i,j} = -\frac{1}{h_{i,j}}\left(c_{i,j+1}\beta_{i,j+1}\right) + (-1)^j \frac{1}{h_{i,j}}\beta'_{i,j}a_{i,1} \qquad N-1 \ge i \ge 3 \quad \text{... (D11)}$$

$$\beta_{\mathrm{i},2} = -\frac{1}{h_{\mathrm{i},2}}\left(c_{\mathrm{i},3}\beta_{\mathrm{i},3}\right) + \frac{1}{h_{\mathrm{i},2}}a_{\mathrm{i},1} \quad \text{... (D12)}$$

$$h_{i,1} = b_{i,1} - c_{i,2}\beta_{i,2} - \alpha_{i,1}\beta_{i,N} \quad \text{... (D13)}$$

[$\mathbf{U_1}$(λ)], [$\mathbf{U_2}$(λ)], and [**L**(λ)] are formed using elements of [**A**] and Eqs (D2)–(D13).

A preliminary solution of $[\mathbf{Q}]$ given as $[\mathbf{Y}]$ is obtained using $[\mathbf{U_1}(\lambda)][\mathbf{Y}] = [\mathbf{B}]$ where $[\mathbf{B}]$ is the RHS member of Eq. (17).

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & f_{i,3} & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & f_{i,N-2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & f_{i,N-1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & f_{i,N} & 1 \end{bmatrix} \begin{bmatrix} Y_{i,1}^{(n+1)} \\ Y_{i,2}^{(n+1)} \\ Y_{i,3}^{(n+1)} \\ \vdots \\ Y_{i,N-2}^{(n+1)} \\ Y_{i,N-1}^{(n+1)} \\ Y_{i,N}^{(n+1)} \end{bmatrix} = \begin{bmatrix} B_{i,1} \\ B_{i,2} \\ B_{i,3} \\ \vdots \\ B_{i,N-2} \\ B_{i,N-1} \\ B_{i,N} \end{bmatrix} \quad \text{... (D14)}$$

From Eq. (D14),

$$Y_{i,1}^{(n+1)} = B_{\mathrm{i},1} \quad \text{... (D15)}$$

$$Y_{i,2}^{(n+1)} = B_{\mathrm{i},2} \quad \text{... (D16)}$$

$$Y_{i,j}^{(n+1)} = B_{i,j} - f_{i,j} Y_{i,j-1}^{(n+1)} \qquad 3 \le j \le N \quad \text{... (D17)}$$

An intermediate solution of $[\mathbf{Q}]$ given as $[\mathbf{Z}]$ is obtained using $[\mathbf{U_2}(\lambda)][\mathbf{Z}] = [\mathbf{Y}]$.

$$\begin{bmatrix} h_{i,1} & c_{i,2} & 0 & 0 & \dots & 0 & 0 & 0 & a_{i,N} \\ 0 & h_{i,2} & c_{i,3} & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{i,3} & c_{i,4} & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & h_{i,N-2} & c_{i,N-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & h_{i,N-1} & c_{i,N} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & h_{i,N} \end{bmatrix} \begin{bmatrix} Z_{i,1}^{(n+1)} \\ Z_{i,2}^{(n+1)} \\ Z_{i,3}^{(n+1)} \\ \vdots \\ Z_{i,N-2}^{(n+1)} \\ Z_{i,N-1}^{(n+1)} \\ Z_{i,N}^{(n+1)} \end{bmatrix} = \begin{bmatrix} Y_{i,1}^{(n+1)} \\ Y_{i,2}^{(n+1)} \\ Y_{i,3}^{(n+1)} \\ \vdots \\ Y_{i,N-2}^{(n+1)} \\ Y_{i,N-1}^{(n+1)} \\ Y_{i,N}^{(n+1)} \end{bmatrix}$$

which gives

$$Z_{i,N}^{(n+1)} = \frac{Y_{i,N}^{(n+1)}}{h_{i,N}} \quad \text{... (D18)}$$

$$Z_{i,j}^{(n+1)} = \frac{1}{h_{i,j}} \left(Y_{i,j}^{(n+1)} - c_{i,j+1} Z_{i,j+1}^{(n+1)} \right) \qquad N-1 \ge j \ge 2 \quad \text{... (D19)}$$

$$Z_{i,1}^{(n+1)} = \frac{1}{h_{i,1}} \left(Y_{i,1}^{(n+1)} - c_{i,2} Z_{i,2}^{(n+1)} - a_{i,N} Z_{i,N}^{(n+1)} \right) \quad \text{... (D20)}$$

The solution of $[\mathbf{Q}]$ is obtained using $[\mathbf{L}(\lambda)][\mathbf{Q}] = [\mathbf{Z}]$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \beta_{i,2} & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \beta_{i,3} & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \beta_{i,N-2} & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ \beta_{i,N-1} & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ \beta_{i,N} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_{i,1}^{(n+1)} \\ Q_{i,2}^{(n+1)} \\ Q_{i,3}^{(n+1)} \\ \vdots \\ Q_{i,N-2}^{(n+1)} \\ Q_{i,N-1}^{(n+1)} \\ Q_{i,N}^{(n+1)} \end{bmatrix} = \begin{bmatrix} Z_{i,1}^{(n+1)} \\ Z_{i,2}^{(n+1)} \\ Z_{i,3}^{(n+1)} \\ \vdots \\ Z_{i,N-2}^{(n+1)} \\ Z_{i,N-1}^{(n+1)} \\ Z_{i,N}^{(n+1)} \end{bmatrix} \quad \dots \text{(D21)}$$

From Eq. (D21),

$$Q_{i,1}^{(n+1)} = Z_{i,1}^{(n+1)} \quad \dots \text{(D22)}$$

$$Q_{i,j}^{(n+1)} = Z_{i,j}^{(n+1)} - \beta_{i,j} Q_{i,1}^{(n+1)} \qquad 2 \le i \le N \quad \dots \text{(D23)}$$

Equations (D22) and (D23) give the solution $Q_{i,\mathrm{j}}^{(n+1)}$ of Eq. (17) for $1 \le j \le N$.