

Chaotic Characteristics of RN Black Holes

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Received 26 August 2024; accepted 27 January 2025

The nonlinear characteristics of general relativity are manifested in the chaotic behavior of black holes. Foot points of chaos can give an insight to understanding black hole phenomena. This paper aims to identify and analyze the chaotic behavior of Reissner Nordstrom(RN) black holes. A trajectory analysis of a test particle near Reissner Nordstrom metric performed using the Euler Lagrangian and action angle variable methods. The presence of homoclinic orbit implies the existence of chaos in dynamical system. We have evaluated the cutoff radial distance for the onset of chaotic motion for Reissner Nordstrom black holes metric with different charges. We have shown that geodesic motion around RN black holes with different charge values may become chaotic. A comparative analysis of the effective potential for Schwarzschild and RN black holes, along with an examination of homoclinic orbit, is presented in this paper. Therefore, we can say that these findings can lead to astrophysical application in many body relativistic models.

Keywords: Black holes; Melnikov method; Chaos; Periodic Perturbations; Homoclinic Orbit; Geodesic Motion

1 Introduction

Chaotic behavior is a widespread feature of relativistic systems, including cosmological models of the universe. The Solar system dynamics also has a chaotic feature which was studied by Poincare in his article “the problem in three bodies and the equations”¹. A major stumbling block in the study of chaotic behavior in gravitating systems is the difficulty in finding unambiguous signatures of chaos. In the case of geodesic motion, there is a classic proof that motion in spaces of negative curvature is chaotic². Chaotic behavior gives us idea about the various nonlinear incidents in the general relativity. The identification of the temporal behavior of the black hole system as chaotic has a new window towards the understanding of the origin and nature of their variability. The present study can be extended to characterize the chaotic behavior³. Lei *et al.*⁴ describe the chaotic behavior of black hole. This paper introduces the black hole chaos and finds some chaos indicators from static equilibrium with the help of toy model. This preprint give us idea about how can we describe chaos near Reissner-Nordstrom black hole. Studying chaos around the other black hole can help us to understand the characteristics of black hole. Bombelli & Calzetta¹ describe the chaotic behavior of Schwarzschild Metric using Melnikov method.

Letelier & Vieira⁵ have shown the chaotic behavior of test particles moving around Schwarzschild black hole using Melnikov method. Recent series of papers of Polcar & Semerak⁶ used the Melnikov method to detect the homoclinic orbit of the Schwarzschild black hole simulated by the Nowak-Wagoner pseudo-potential and encircled by the Bach-Weyl ring or the inverted first Morgan-Morgan disc, and the extreme Reissner-Nordstrom black hole encircled by the extremely charged, Ring *et al.*⁷ briefly described the chaotic motion around a black hole under minimal length effects about the Schwarzschild metric as a special case. These papers give us an idea about how we can describe chaos near rest metrics of black hole. Zhou *et al.*⁸ explained thermodynamical properties of RN-AdS black hole immersed in perfect Fluid Dark Matter. Lei *et al.*⁹ revealed a link between chaos bound violations and thermodynamic stability. Gonzalez *et al.*¹⁰, explored the time like geometry of five-dimensional Schwarzschild and Reissner Nordstrom anti-de Sitter black hole, shedding light on their motion and orbit properties.

Motion around the black hole creates an orbit and this orbit behaves like homoclinic orbit. Therefore we can say that formation of homoclinic orbit around black hole is the first step towards recognizing chaotic behavior. In this paper we investigate the chaotic characteristics of black hole using semi-classical method. Here, we have obtained expressions of effective potential and homoclinic orbit Reissner

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Nordstrom black hole metrics. A detailed analysis of test particle motion along the vicinity of Reissner Nordstrom black hole utilizing the standard metric in Boyer Lindquist coordinates system has been done. Using the application of Euler-Lagrangian and action variables methods, we determine the expressions for effective potential and radial momentum and discuss the implications of our findings.

This paper is organized as follows: in section 2 we have described Melnikov method and we have evaluated effective potential of different metrics of black hole. Here, we have described and obtained expression for radial momentum of blackhole metrics. Section 3 is dedicated to results and discussion. Section 4 is described the conclusion of this work. The last sections 5 and 6 provide acknowledgement and references of this paper.

2 Theory and Method

2.1 Study of Melnikov Function

Homoclinic orbit is a closed trajectory which has a hyperbolic saddle fixed point of the respective flow as both its past and future asymptote. Along such an orbit, the tangent bundle of the configuration manifold splits into stable and unstable invariant sub-bundles, spanning sub-manifolds in which the flow is contracting and expanding, respectively (thus saddle point). In the unperturbed system, these sub manifolds intersect “longitudinally” and thus coincides along the Homoclinic orbit. If the perturbation deform them in such a way that they start to intersect transversally along that orbit, it is a signature that the orbit has “broken up” into a chaotic layer (so-called Smale-Birkhoff Homoclinic theorem)⁶.

In the case of a Hamiltonian perturbation, the Melnikov integral (I) is given by

$$I(t_0) = \int_{-a}^a dt \{H_0, G\} \quad \dots (1)$$

H_0 is the initial Hamiltonian and the perturbation G is a periodic function of time, with period T .

Where the integral is taken along the unperturbed Homoclinic loop L , and

$\{.,.\}$ stands for the Poisson bracket and (q, p) are canonical coordinates in phase space.

$$\{f, g\} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} \quad \dots (2)$$

Equation (1) is the form under which we will use the Melnikov method to study chaotic behavior under perturbations of a black-hole space time¹. When

Melnikov integral shows isolated zeros, then the homoclinic orbits will split and transversal cuts will appear under the perturbation. Therefore there will be Smale horseshoes invariant in the perturbed phase space. Melnikov integral we require the existence of homoclinic orbits¹.

2.2 Hamiltonian Trajectory

While describing Homoclinic trajectory around black hole we have considered a relativistic particle moving in a x space time described by the metric g_{ab} . The world-line of a particle will be denoted by $x^a(s)$ and its mass by μ . The motion of the particle can be obtained from the action¹

$$S[x] = \frac{\mu}{2} \int g_{ab} x^a \dot{x}^b ds \quad \dots (3)$$

This action is not reparametrization invariant, but is made simple for our purposes is the proper time along the world-line. The canonical conjugate momentum to x^a is $P_a = \mu g_{ab} \dot{x}^b$ and satisfy the mass shell constraint.

$$g^{ab} P_a P_b = -\mu^2 \quad \dots (4)$$

The Hamiltonian of the system is given by

$$H = \frac{1}{2\mu} g^{ab} P_a P_b \quad \dots (5)$$

We take the metric of different black hole to find the homoclinic path of black hole. We have

$$E = -P_t = \mu f \frac{dt}{ds} \quad \dots (6)$$

$$L = P_\phi = \mu r^2 \sin^2 \theta \frac{d\phi}{ds} \quad \dots (7)$$

Using the value of E and L and help of above equations we get the equation of motion

$$\mu^2 \left(\frac{dr}{ds}\right)^2 + f(r) \left(\mu^2 + \frac{L^2}{r^2}\right) = E^2 \quad \dots (8)$$

$$p_r = \frac{\mu}{f} \frac{dr}{ds} \quad \dots (9)$$

Using the above equations for different metrics of Reissner Nordstrom black hole we get the effective potential and radial momentum for black hole.

2.3 Effective potential for Schwarzschild Metric

Schwarzschild black hole solution metric is given below^{13,21}

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \dots (10)$$

Where $f = \left(1 - \frac{2M}{r}\right)$

For existence of Homoclinic orbit the potential for Schwarzschild metric is given $V(r)$. $V(r)$ has an unstable equilibrium point.

$$V(r) = -\frac{2M\mu^2}{r} - \frac{2ML^2}{r^3} + \frac{L^2}{r^2} \quad \dots (11a)$$

Potential $V(r)$ has extremum at real values of x if $\frac{12M^2\mu^2}{L^2} < 1$ and the extreme values are located at r_{stable} and $r_{unstable} = \frac{1}{3}(1 - \Delta)$ and $= \frac{1}{3}(1 + \Delta)$, where

$$\Delta = \sqrt{1 - \frac{12M^2\mu^2}{L^2}}$$

We take $M=1, \mu = 10^{-6}$ and $L = 3.78 \times 10^{-6}$. then $\Delta = 0.4$. We take above assumption because at this value test particle formed homoclinic orbit around the Schwarzschild black hole metrics^{1,5}.

We can write the above effective potential in terms of Δ

$$V(r) = -\frac{L^2}{6Mr}(1 - \Delta^2) - \frac{2ML^2}{r^3} + \frac{L^2}{r^2} \quad \dots (11b)$$

2.4 Effective potential for Reissner Nordstrom Metric

In this section we determine the Effective Potential for Reissner Nordstrom black hole which metric is given by^{13,21}

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \dots (12)$$

where Q =charge

r = radial distance from black hole (r is multiple of R_{Sch}).

M =mass of black hole (M is in multiples of Sun mass)

$$\text{and } f = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)$$

Let $\theta = \pi/2, d\theta = 0$ and μ =mass of test particle or moving particle.

Using above assumptions and Eqs (4) to (7) we get the following expression

$$\mu^2 \left(\frac{dr}{ds}\right)^2 - \frac{2M\mu^2}{r} - \frac{2ML^2}{r^3} + \frac{L^2}{r^2} + \frac{Q^2L^2}{r^4} + \frac{Q^2\mu^2}{r^2} = (E^2 - \mu^2) \quad \dots (13)$$

Comparing above equation with general equation we get

$$V(r) = -\frac{2M\mu^2}{r} - \frac{2ML^2}{r^3} + \frac{L^2}{r^2} + \frac{Q^2L^2}{r^4} + \frac{Q^2\mu^2}{r^2} \quad \dots (14)$$

We can write the above effective potential in terms of Δ

$$V(r) = -\frac{L^2}{6Mr}(1 - \Delta^2) - \frac{2ML^2}{r^3} + \frac{L^2}{r^2} + \frac{Q^2L^2}{r^4} + \frac{Q^2\mu^2}{12M^2r^2}(1 - \Delta^2) \quad \dots (15)$$

Here, we also take $M=1, \mu = 10^{-6}$ and $L = 3.78 \times 10^{-6}$. for $\Delta = 0.4$. We found the homoclinic orbit of test particles around Reissner Nordstrom black hole metric.

The graph between $V(r)$ (Effective potential) and r has been generated for different values of charge Q which are shown in the Figs 1(a) to 11(a).

2.5 Homoclinic Orbit of Black Hole

Homoclinic orbits have been introduced by Poincare' more than a century ago, and since then, they became a fundamental tool in the study of chaos. Poincare had observed that the fixed points of some functions possess an attracting and a repelling direction. This means there is a curve of points moving toward the fixed point, like a vein returning blood to the heart, and a curve of points moving away, like an artery sending blood into the body. If these curves cross, the points of intersection, called Homoclinic points.

Using Eqs (8), (9) and (13) we get expressions for radial momentum (p_r)

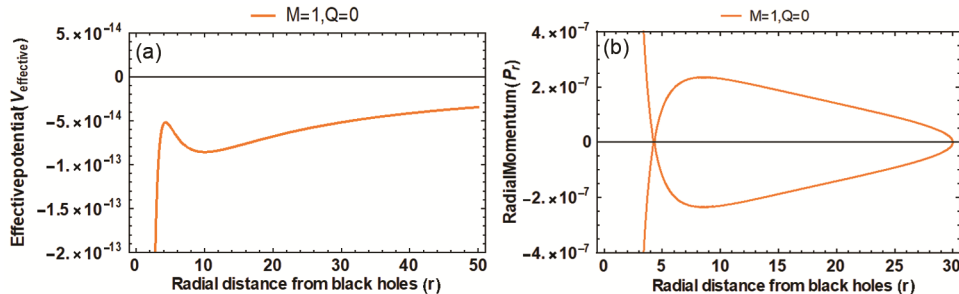


Fig. 1 (a) — Plot of Effective potential (Schwarzschild black hole metric) (b) Plot of Homoclinic orbit trajectory in phase space with energy $0.97372891 \times 10^{-6}$

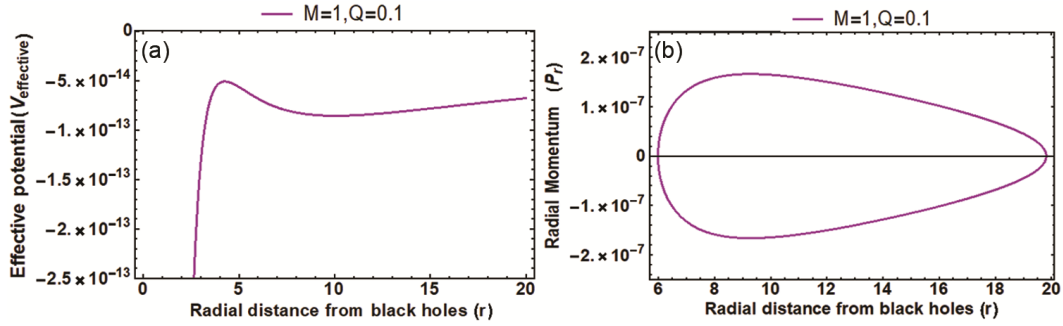


Fig. 2(a) — Plot of Effective potential with charge $Q=0.1$ (b) Plot of Homoclinic orbit trajectory in phase space with charge $Q=0.1$ and energy $E = 0.965302 \times 10^{-6}$

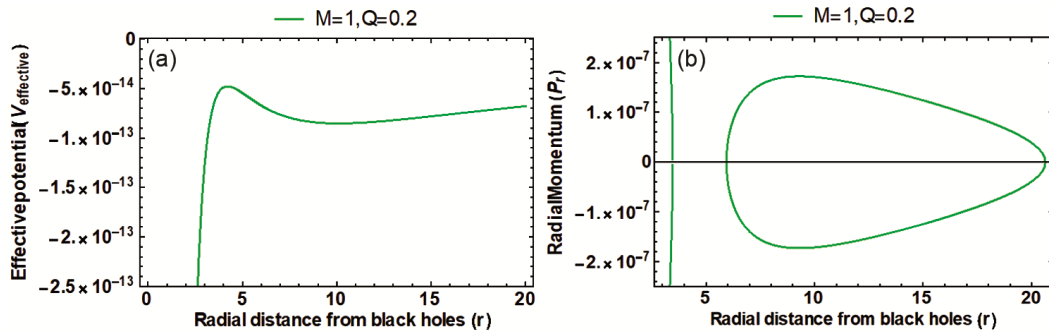


Fig. 3(a) — Plot of Effective potential with charge $Q=0.2$ (b) Plot of Homoclinic orbit trajectory in phase space with charge $Q=0.2$ and energy $E = 0.9661594 \times 10^{-6}$

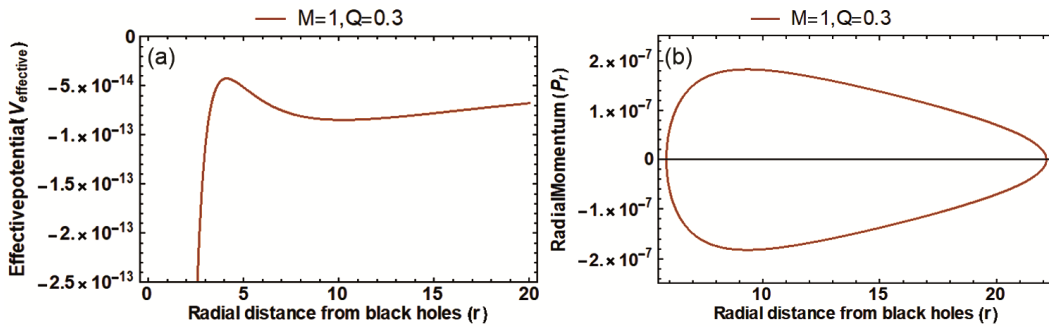


Fig. 4(a) — Plot of Effective potential with charge $Q=0.3$ (b) Plot of Homoclinic orbit trajectory in phase space with charge $Q=0.3$ and energy $E = 0.96765438 \times 10^{-6}$

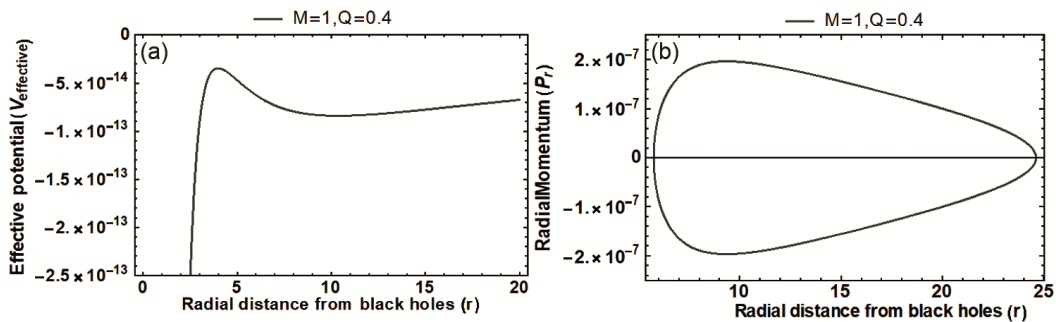


Fig. 5(a) — Plot of Effective potential with charge $Q=0.4$ (b) Plot of Homoclinic orbit trajectory in phase space with charge $Q=0.4$ and energy $E = 0.969884 \times 10^{-6}$

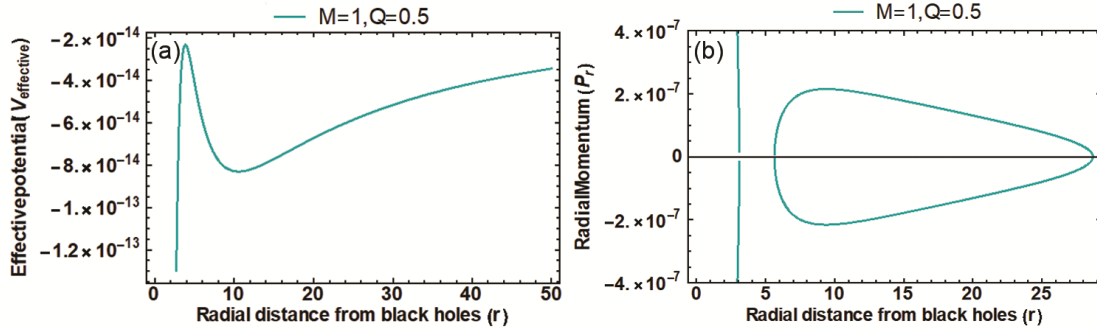
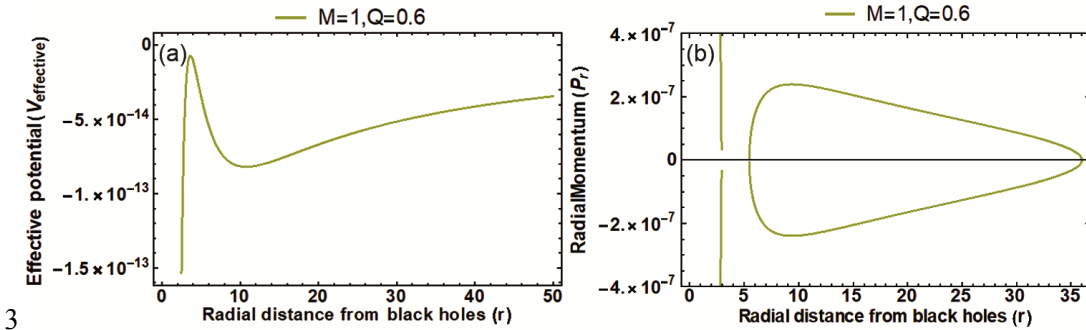


Fig. 6(a) — Plot of Effective potential with charge $Q=0.5$ (b) Plot of Homoclinic orbit trajectory in phase space with charge $Q=0.5$ and energy $E=0.9730164 \times 10^{-6}$



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Fig. 7(a) — Plot of Effective potential with charge $Q=0.6$ (b) Plot of Homoclinic orbit trajectory in phase space with charge $Q=0.6$ and energy $E=0.97731315 \times 10^{-6}$

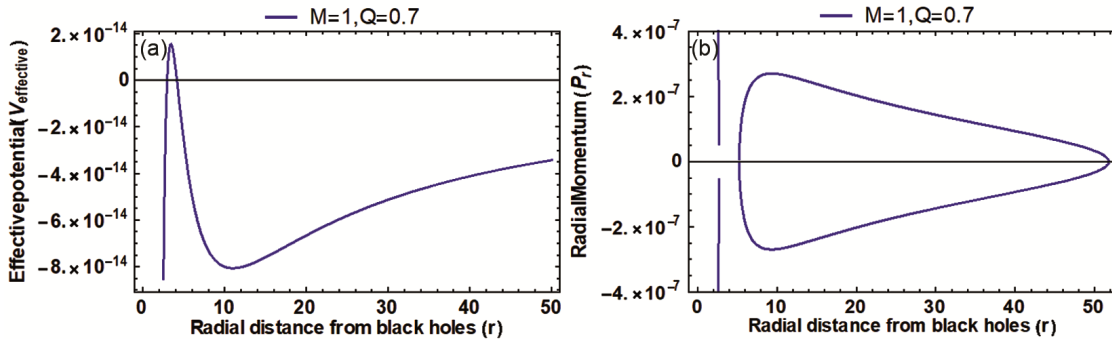


Fig. 8(a) — Plot of Effective potential with charge $Q=0.7$ (b) Plot of Homoclinic orbit trajectory in phase space with charge $Q=0.7$ and energy $E=0.983198 \times 10^{-6}$

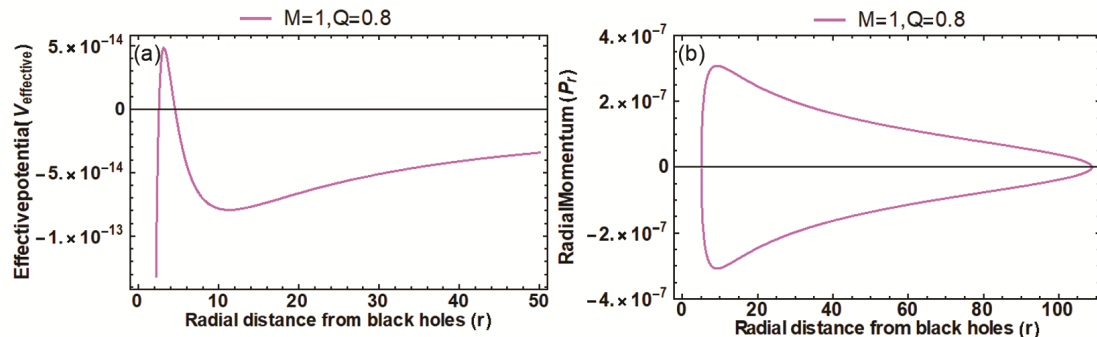


Fig. 9(a) — Plot of Effective potential with charge $Q=0.8$ (b) Plot of Homoclinic orbit trajectory in phase space with charge $Q=0.8$ and energy $E=0.9913889 \times 10^{-6}$

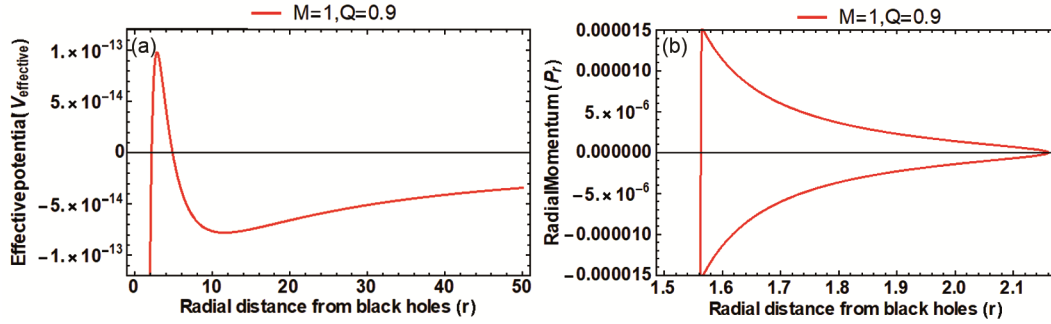


Fig. 10(a) — Plot of Effective potential with charge Q=0.9 (b) Plot of Homoclinic orbit trajectory in phase space with charge Q=0.9 and energy $E= 1.0032248 \times 10^{-6}$

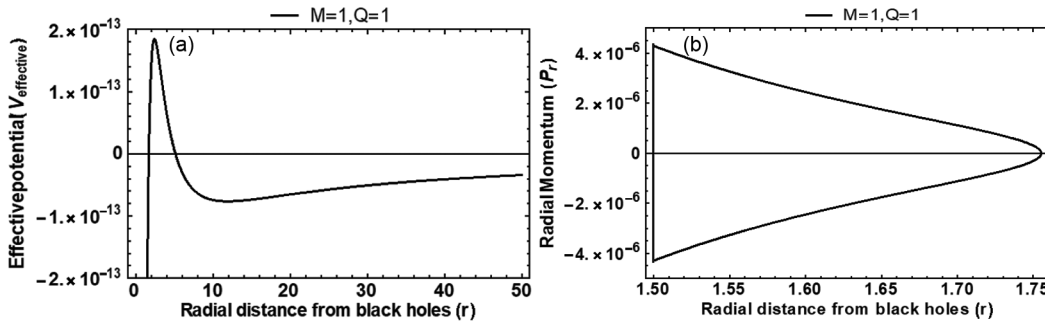


Fig. 11(a) — Plot of Effective potential with charge Q=1.0 (b) Plot of Homoclinic orbit trajectory in phase space with charge Q=1.0 and energy $E= 1.02165 \times 10^{-6}$

$$p_r = \frac{\mu}{f} \frac{dr}{ds} = \pm \frac{1}{f} \left[E^2 - f \left(\mu^2 + \frac{L^2}{r^2} \right) \right]^{1/2} \quad \dots (16)$$

Where f = Metric of black hole.
 r = Radius of black hole.
 E = Energy,
 L = Angular momentum

μ = Relative mass of particle moving around black hole.

$$p_r = \frac{\pm 1}{(1 - \frac{2M}{r})} \sqrt{ \left(E^2 - u^2 - \frac{L^2}{r^2} + 2 \times M \times \frac{1}{r} \times u^2 + \frac{2ML^2}{r^3} \right) } \quad \dots (17)$$

$$p_r = \frac{\pm 1}{(1 - \frac{2M}{r} + \frac{Q^2}{r^2})} \sqrt{ \left(E^2 - u^2 - \frac{L^2}{r^2} - u^2 \times \frac{Q^2}{r^2} - L^2 \times \frac{Q^2}{r^4} + 2 \times M \times \frac{1}{r} \times u^2 + \frac{2ML^2}{r^3} \right) } \quad \dots (18)$$

With the help of above equation we have evaluated the radial momentum of different types of black hole metrics. We have also plotted the graphs between radial momentum (P_r) and r as shown in Figs 1(b) to 11(b). These graphs represent the homoclinic orbit of black hole. Therefore we can infer that the existence homoclinic orbit is the first step towards the chaotic motion around the black hole.

3 Results and Discussion

In this study we have investigated the motion of particle around the Reissner Nordstrom black hole metric. We evaluated the expressions for effective potential and radial momentum of moving particle around Reissner Nordstrom black hole metric. A comparative analysis of effective potential and homoclinic orbits with variable charge values are also analyzed in this study.

Here, we have assumed some values for graphical analysis of Homoclinic orbit in phase space. We take $M=1$, $\mu = 10^{-6}$ and $L = 3.78 \times 10^{-6}$ which correspond to $\Delta=0.4$. For Schwarzschild black hole energy $E = 0.97372891 \times 10^{-6}$. But for Reissner Nordstrom black hole metrics energy changes with charge Q . Using Eqs (11) and (14) we calculated the effective potential for different charge. Eqs (11) and (14) contain an attractive term ($1/r$), a repulsive term ($\frac{1}{r^2}$) and an additional term ($\frac{1}{r^3}$) and this additional term is responsible for homoclinic orbits.

Figures 1(a) to 11(a) represent the effective potential variation with charges and radial distances. These graphical analyses provided the unstable

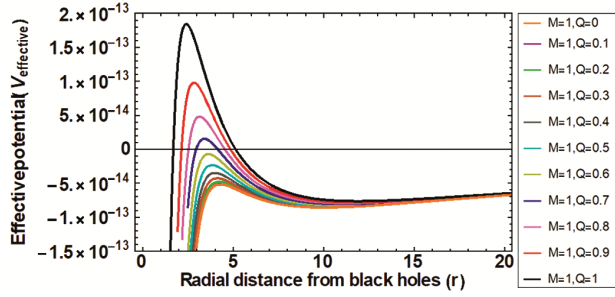


Fig. 12 — Plot of effective potential with increasing charge values $Q=0.0$ to $Q=1.0$ and different energies values

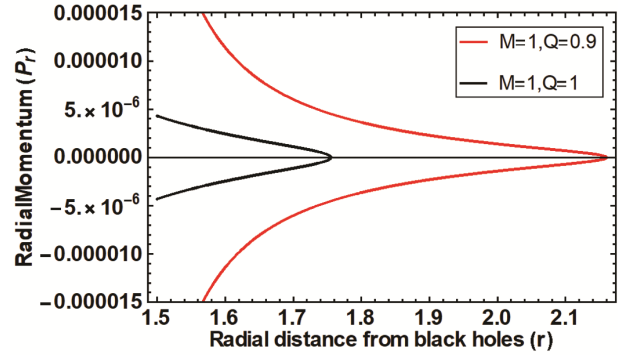


Fig. 14 — Plot of Homoclinic orbit trajectory with increasing charge values $Q=0.9$ and $Q=1.0$ with different energies values

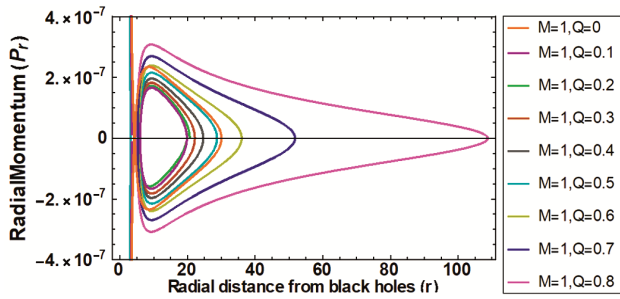


Fig. 13 — Plot of Homoclinic orbit trajectory in phase space with increasing charge values $Q=0.0$ to $Q=0.8$ and energies values, and different energies values

Nordstrom black hole metric. The radial momentum has positive and negative signs which correspond to the two parts of the orbit. Therefore we can say that plot between radial momentum and radial distance represents the homoclinic orbit in phase space. These graphs provide the cut off radial distance for moving particles around Reissner Nordstrom black hole metric. Here, we can say that if radial distance is less than cut off radial distance then the orbit of the moving test particle diverges and do not show chaotic nature of motion around Reissner Nordstrom black hole.

Table. 1 contains the value of energy per unit mass, radial distance for local maximum radial distance and cut off radial distances.

Figures (12-14) provide the comparative analyses of the effective potential and homoclinic orbits with variable charges. From Fig. (12) we have observed the effective potential maximum value increase with the charge Q . Similarly, Figs (13) and (14) also represent the comparative analysis of homoclinic orbits with different charges and energies. From these two graphs we have observed the region of homoclinic orbits increases with charge.

Table-1 — Energy, radial distance for unstable orbits and cut off distance in unit of R_{Sch} with varying charge

Charge(Q)	Energy($\times 10^{-6}$)	Radial distance for unstable orbits	Cut off radial distance
0.0	0.97372891	4.31102	4.32345
0.1	0.965302	4.18098	5.98975
0.2	0.9661594	4.10907	5.93522
0.3	0.96765438	4.03279	5.86877
0.4	0.969884	3.98194	5.77496
0.5	0.9730164	3.78918	5.64984
0.6	0.97731315	3.66205	5.46758
0.7	0.983198	3.36168	5.28079
0.8	0.9913889	3.17563	5.00952
0.9	1.0032248	2.80351	1.56646
1.0	1.02165	2.36938	1.4975

4 Conclusion

In this study our findings are summarized the following points.

- We have evaluated expressions for effective potential and radial momentum of Reissner Nordstrom black hole.
- Presented graphical relationship between the effective potential and radial distance of moving particle around Reissner Nordstrom black hole. We have also found this relationship for different charges ($Q=0$ to $Q=1$).
- Radial distances for unstable orbits are evaluated from the graphs 1(a) to 11(a).

equilibrium for the motion of moving particle around the Reissner Nordstrom black hole metric. The existence of unstable equilibrium is an important indicator showing the existence of homoclinic orbit. We have also found the radial distances from the Reissner black hole at which unstable orbit exists. From Eqs (17), (18) we get Homoclinic orbit of Schwarzschild black hole and Reissner Nordstrom as depicted in figures. Figs 1(b) to 11(b) represent the homoclinic orbits of moving particle around Reissner

➤ A comparative analysis of effective potential with variation of charges also illustrated in figure (12).

➤ Here, we have evaluated the radial momentum for Schwarzschild ($Q=0$) and Reissner Nordstrom ($Q=0.1$ to $Q=1.0$) black hole metric. We have also obtained graphical outputs for the homoclinic orbit of the two metrics of black hole mainly Schwarzschild and Reissner Nordstrom. We have used semi-classical methods to quantifying the cutoff radius for the onset of chaos.

➤ We have calculated the cutoff distance for the onset of chaotic motion for Schwarzschild ($Q=0$) and for Reissner Nordstrom black hole ($Q=0.1$ to $Q=1.0$) respectively.

➤ The region of homoclinic orbits increases with increasing value of charges.

➤ These findings also indicate the how celestial bodies' motion is more sensitive than Newtonian body dynamics. In Newtonian mechanics we see that small changes in variables not more affected the trajectories of moving particle but here we found that the small changes in variable parameters change the trajectory of particle. So we can say that the geodesic motion around Reissner Nordstrom black hole sensitive to initial conditions. Our results will be useful to extend the chaotic characteristics of celestial bodies.

The findings of this study are applicable to the detection of chaos in various black hole metrics.

Acknowledgment

We are thankful to Department of Physics, Patna University and IISER Kolkata (IAGR-32) for

fruitful discussions. We are grateful to Prof. Vijay A. Singh for fruitful discussion.

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